Single-source Shortest Paths

Tao Hou

Problem Definition

Given a weighted (directed or undirected) graph $G = (V, E)$, a ***source*** vertex s and a ***target*** vertex t in G, compute a path from s to t of ***minimum weight*** (i.e., the ***shortest*** path)

- The weight is a function $w : E \longrightarrow \mathbb{R}$ on the edges
- The weight of a path is the sum of weights of all edges on the path

Two variations:

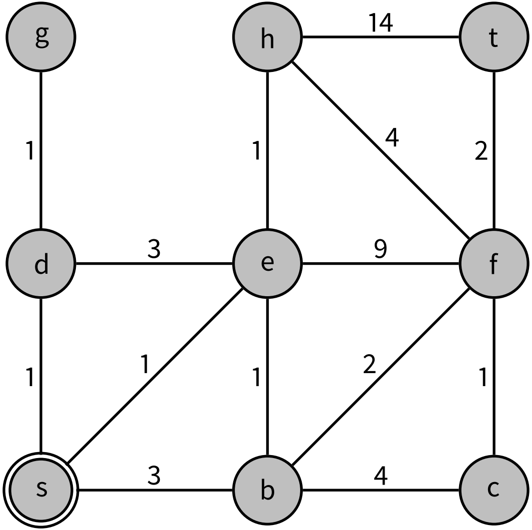
Single-source Shortest Paths

Given a ***source*** vertex s of G, compute the shortest paths from s to ***all other vertices***

All-pair Shortest Paths

Compute the shortest paths for all pairs of vertices

Example



A shortest path from s to t is: $s \longrightarrow e \longrightarrow b \longrightarrow f \longrightarrow t$ of weight 6

- In reality, the weight can be the length, cost, or time of roads, transportation lines etc.

More definitions

- The weight of the shortest path from s to t is called the ***distance***, or ***shortest-path distance***, from s to t and is denoted as $\delta(s, t)$.
- We have $\delta(s, t) = \infty$ if there is no path from s to t

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- We have $\delta(s, t) = \infty$ if there is no path from s to t
- In the problem, edge weights can be ***negative***.
- However, if there is a ***negative-weight*** cycle on the path from s to t, $\delta(s, t)$ (as well as the problem) is not well-defined:
  - We can choose go through the cycles for arbitrary times and the weight of the path can arbitrarily lowered.

Algorithms

- BFS (Review)
- Dijkstra's algorithm
- Bellman-Ford
- An algorithm for DAG

Graph data structures

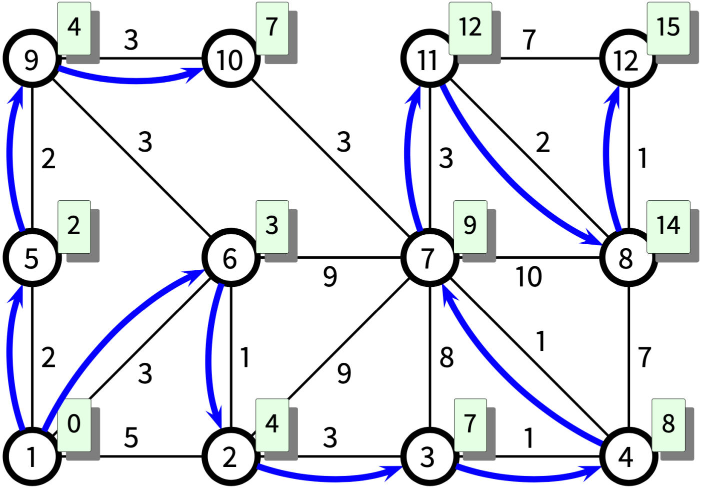
We assume ***adjacency list*** as data structures for graphs

Representing shortest paths from *s*

Through a ***shortest-path tree*** rooted at *s*, where the (unique) simple path from *s* to any vertex *v* in the tree is a shortest path from *s* to *v*

- There must be no cycles in a shortest path
  - The problem is not well-defined with negative-weight cycles
  - There can be no cycles with non-negative weight in a shortest path
- Shortest paths have the ***optimal substructure*** property
- We use $P[v]$ to record the parent of *v* in the tree (like in BFS/DFS)

Example of shortest path tree



Breadth-First Search

- One of the simplest but also a fundamental algorithm
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- *Input:* $G = (V, E)$ and a ***source*** vertex $s \in V$
  - explores the graph starting from s, touching all vertices that are reachable from s
  - computes the distance of each vertex from s ('distance' means minimum number of edges)
  - iterates through the vertices at increasing distance
    - the algorithm discovers all vertices at distance k from s before discovering any vertices at distance $k + 1$ (hence the name)
  - produces a ***BFS tree*** rooted at s
    - An edge (u, v) in the tree means that v is '***discovered***' by visiting u
  - works on both *directed* and *undirected* graphs

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- Breadth-first search computes the single-source shortest paths for s with ***weights of all edges being 1***.
  - The BFS tree is the shortest-path tree in this case

Breadth-First Search: High-level idea

A central data structure: A (FIFO) ***Queue***

Two phases of accessing a vertex u

- ***Discovering***: put u into the queue waiting to be *visited*
- ***Visiting***: access the adjacency list of u and try to *discover* each adjacent vertex

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Coloring for vertices:

- ***white***: 'undiscovered', initial color
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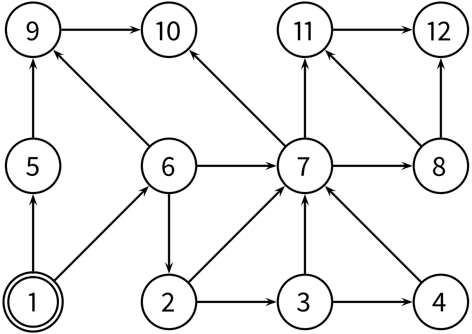
Example



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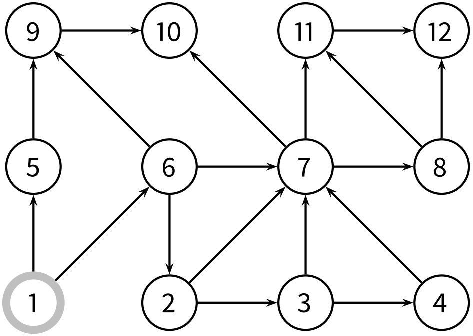

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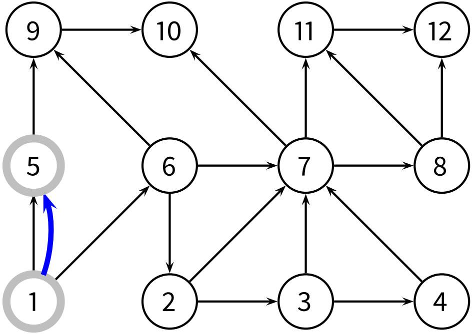

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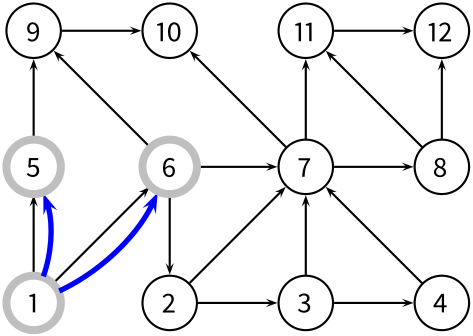

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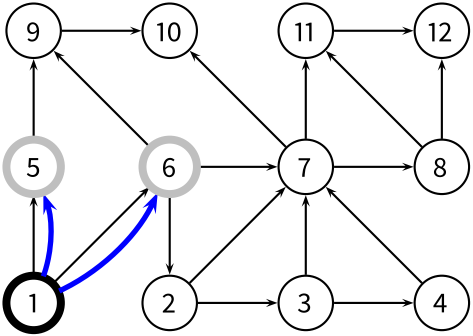

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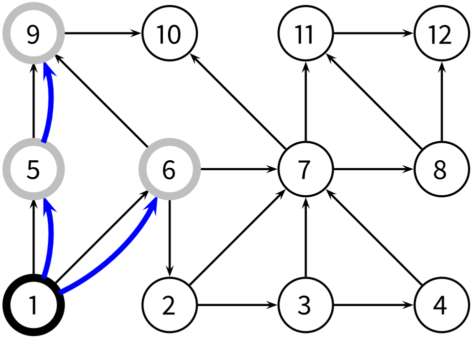

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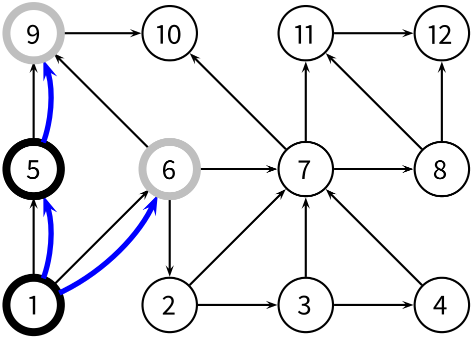

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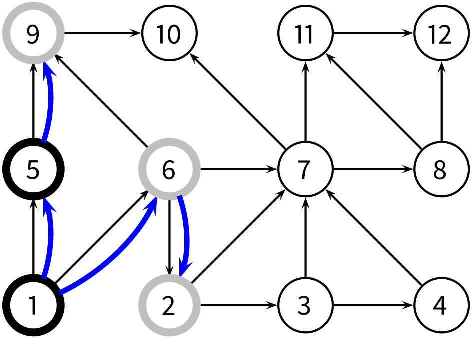

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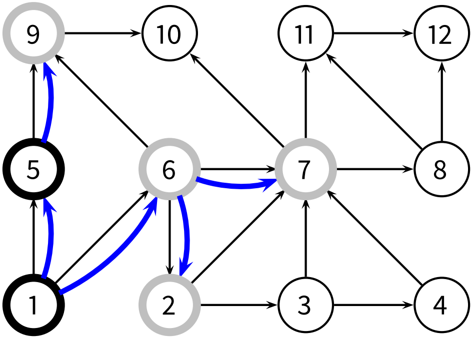

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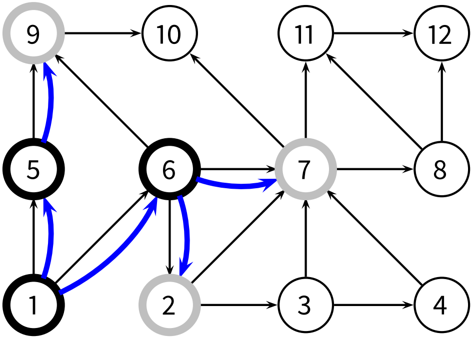

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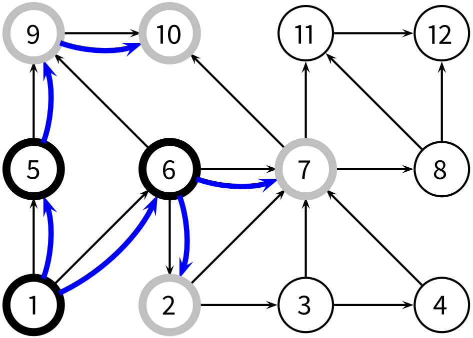

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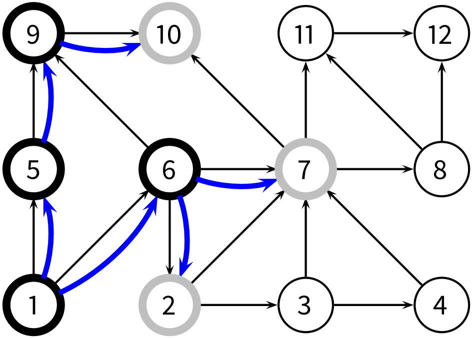

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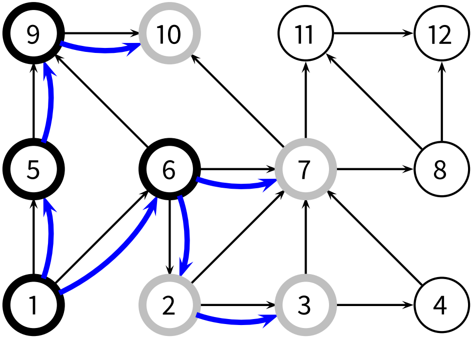

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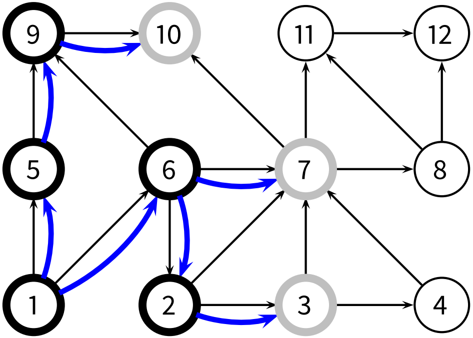

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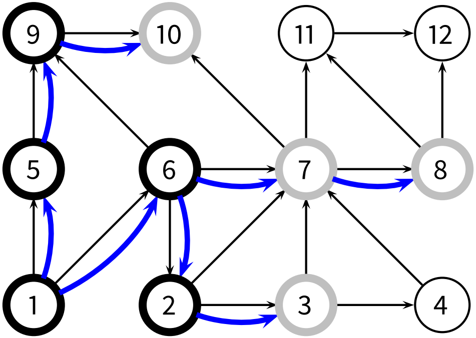

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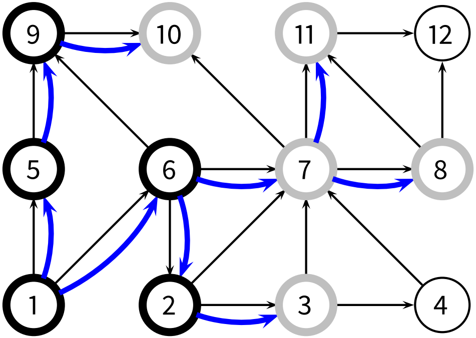

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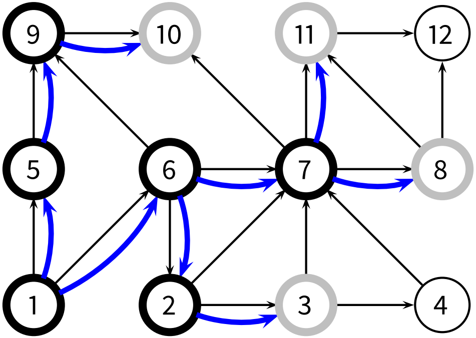

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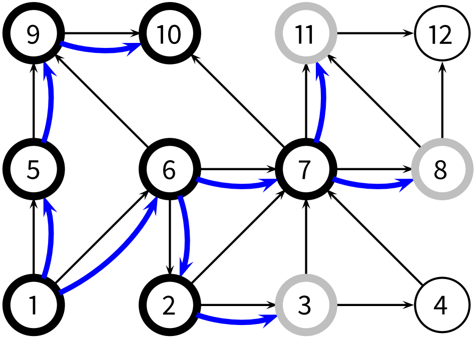

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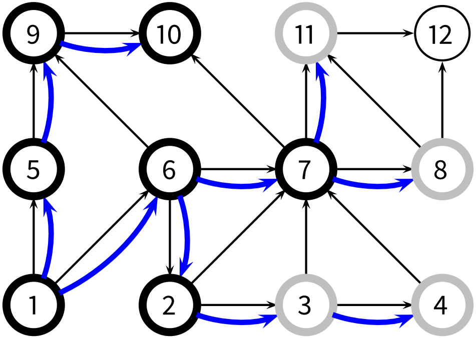

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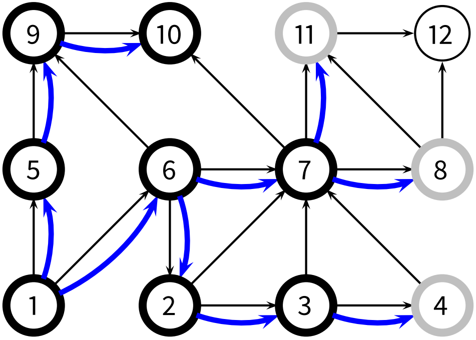

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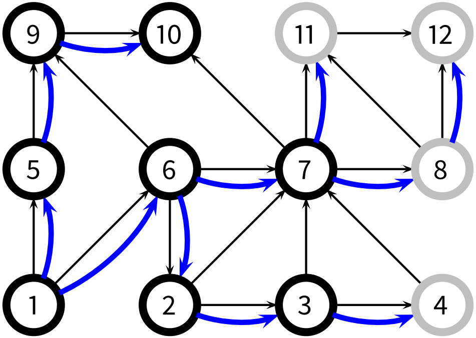

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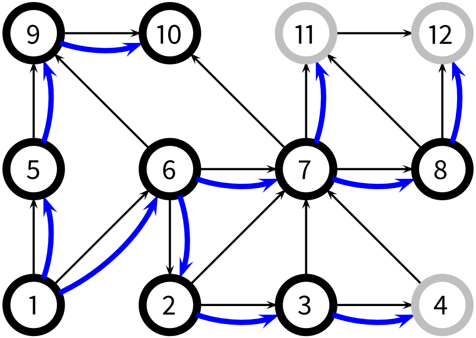

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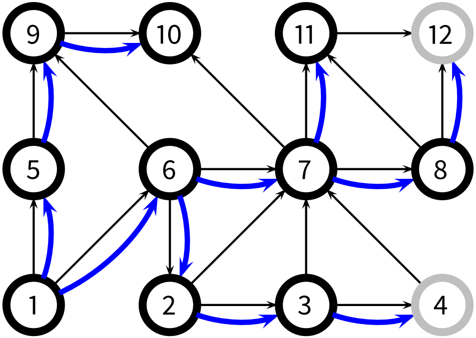

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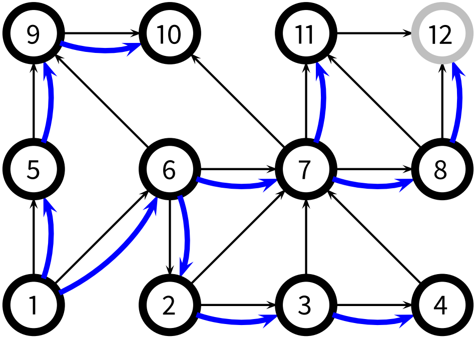

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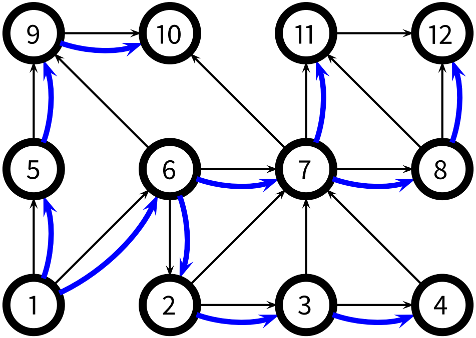

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for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
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-
  - ‘discovering’ means first encountered by the search
  - ‘visiting’ means to try to discover all adjacent vertices which are undiscovered
- Central data structure: a queue (first-in, first-out):
  - Contains ***gray*** vertices
- Some records we keep:
  - $color[u]$: color of a vertex u
  - $d[u]$: distance from s to u
  - $\pi[u]$: a vertex s.t. $(\pi[u], u)$ forms an edge in the BFS tree (there is another interpretation which we will see in Dijkstra’s)

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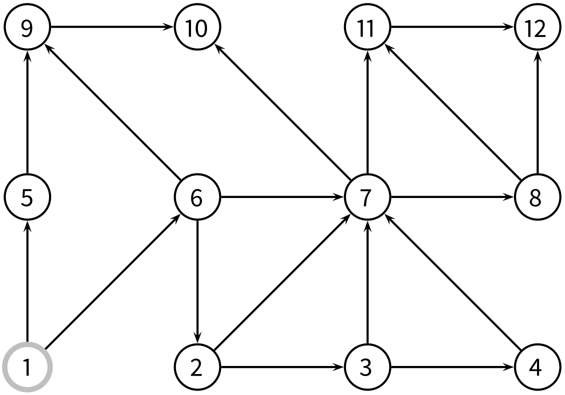

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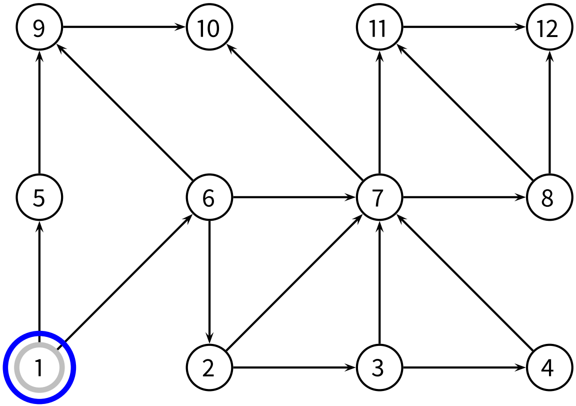

$u = 1$

$Q = \emptyset$

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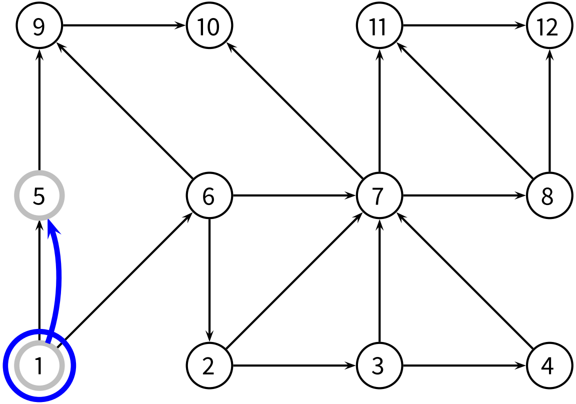

$u = 1$

$Q = \{5\}$

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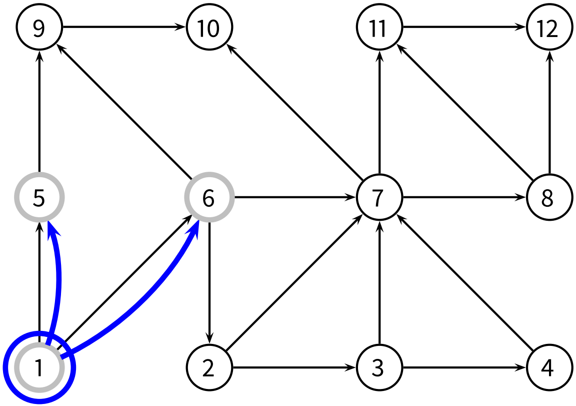

$u = 1$

$Q = \{5, 6\}$

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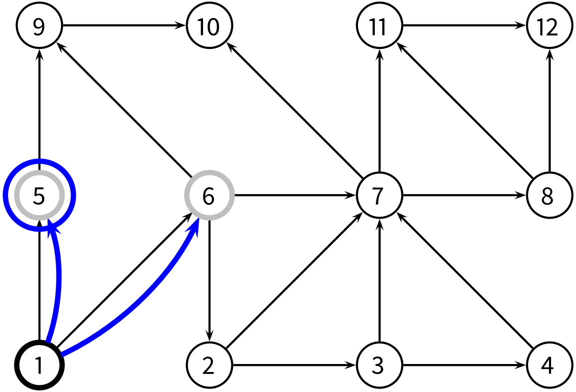

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$Q = \{6\}$

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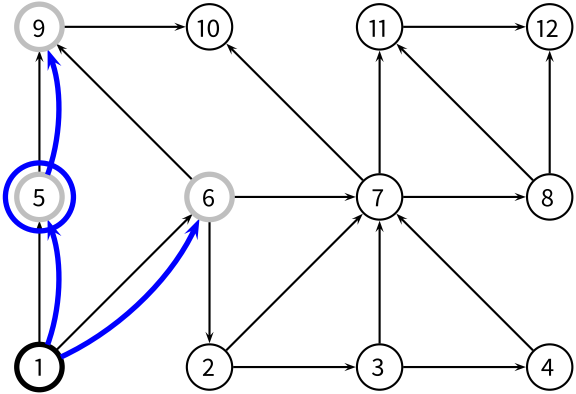

$u = 5$

$Q = \{6, 9\}$

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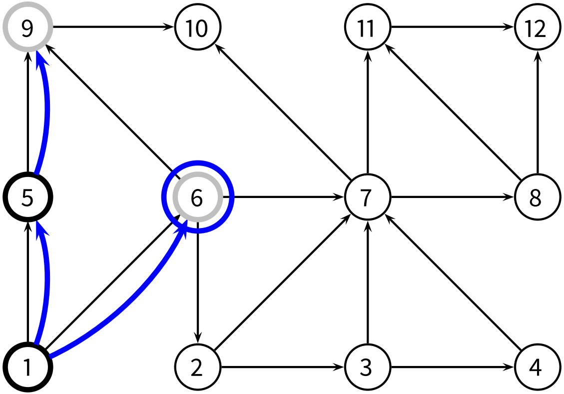
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

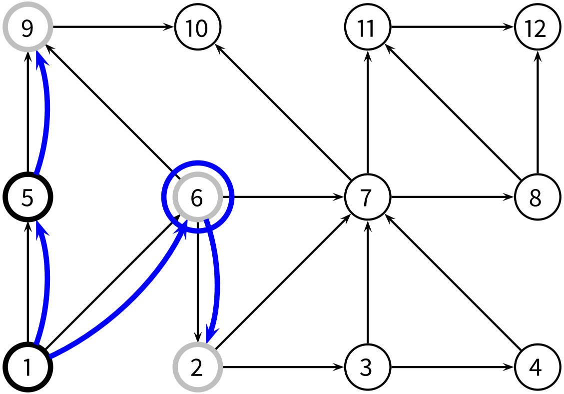


$u = 6$

$Q = \{9\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

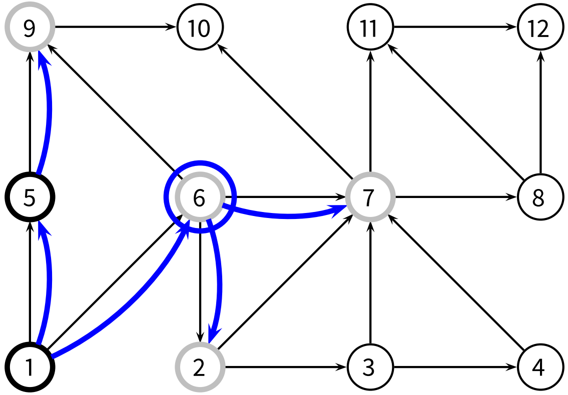


$u = 6$

$Q = \{9, 2, 7\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```



$u = 6$

$Q = \{9, 2, 7\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
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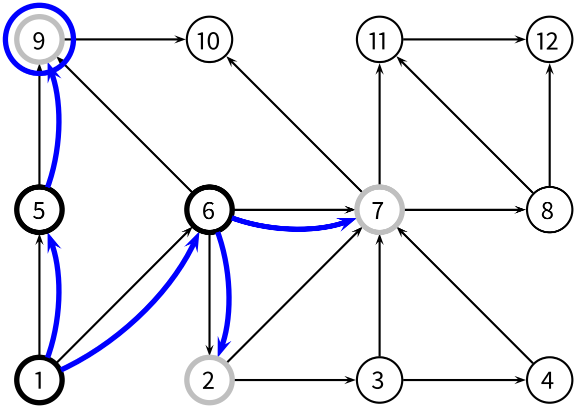
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

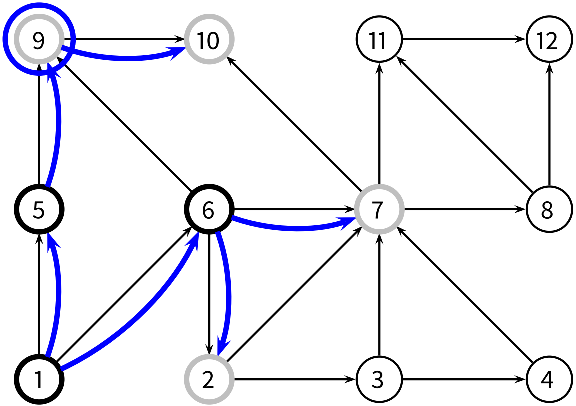


$u = 9$

$Q = \{2, 7\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

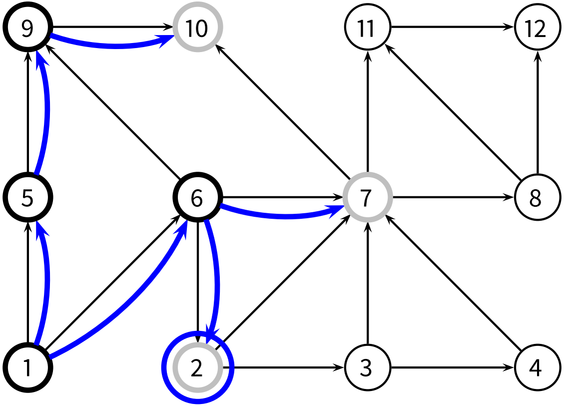


$u = 9$

$Q = \{2, 7, 10\}$

BFS Algorithm

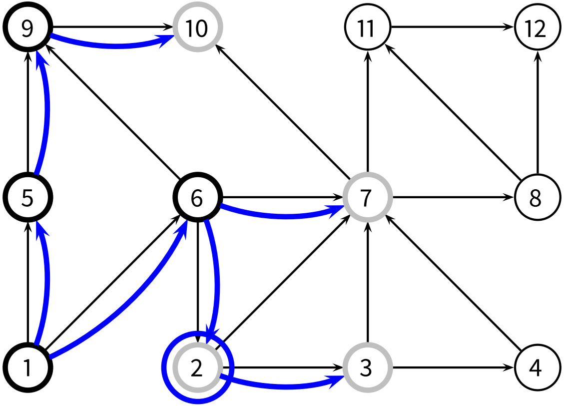
```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



$u = 2$

$Q = \{7, 10\}$

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

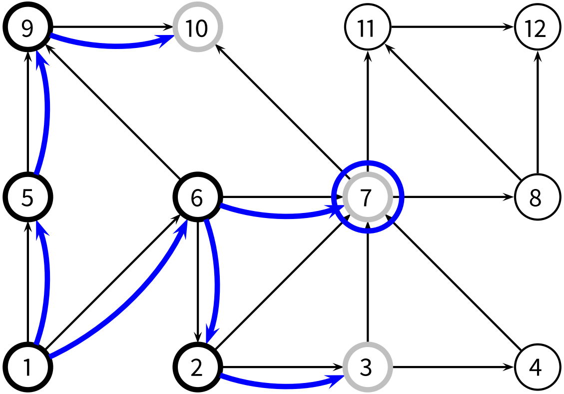


$u = 2$

$Q = \{7, 10, 3\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

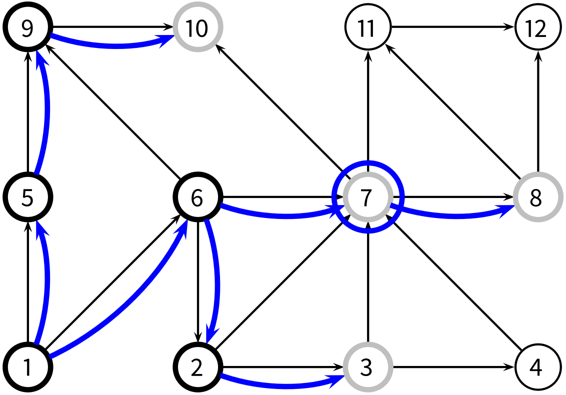


$u = 7$

$Q = \{10, 3\}$

BFS Algorithm

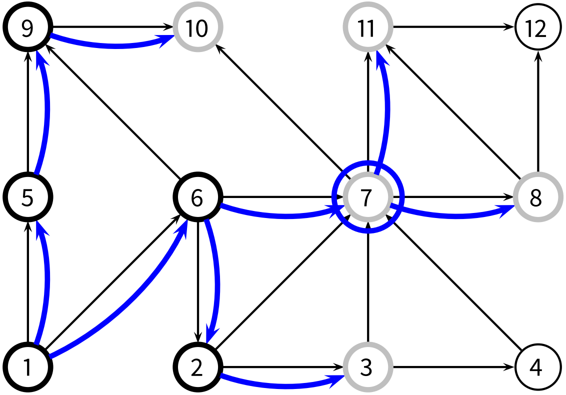
```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



$u = 7$

$Q = \{10, 3, 8\}$

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

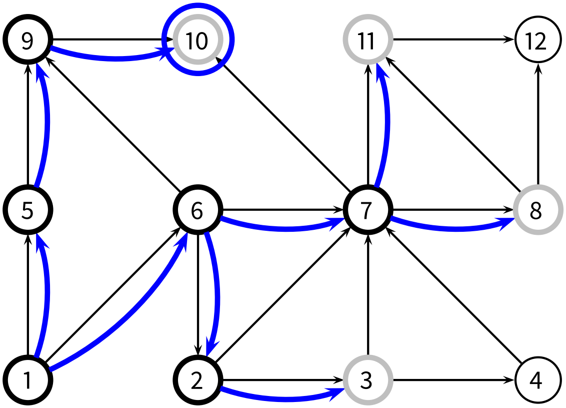


$u = 7$

$Q = \{10, 3, 8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

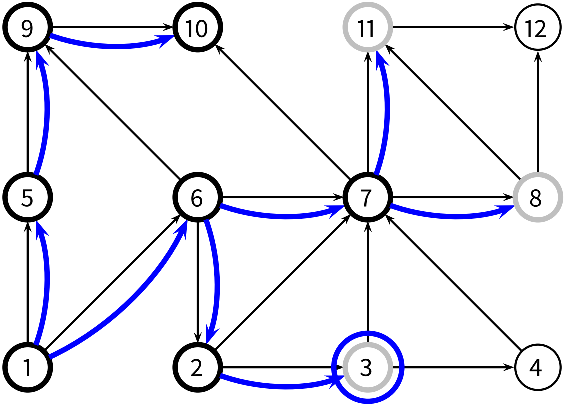


$u = 10$

$Q = \{3, 8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

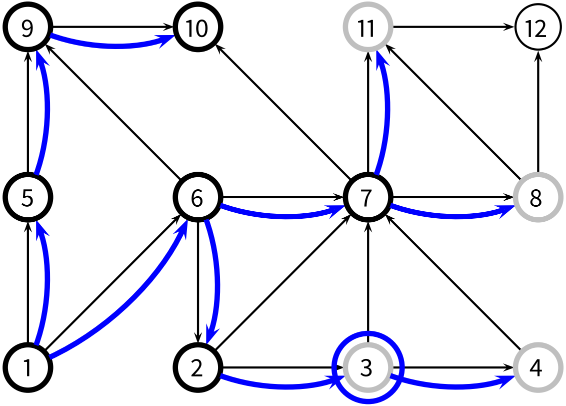


$u = 3$

$Q = \{8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

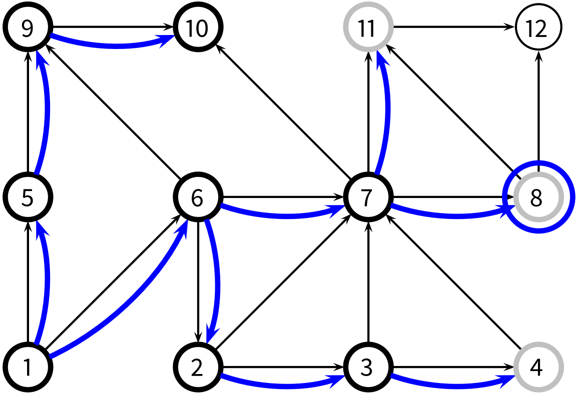


$u = 3$

$Q = \{8, 11, 4\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

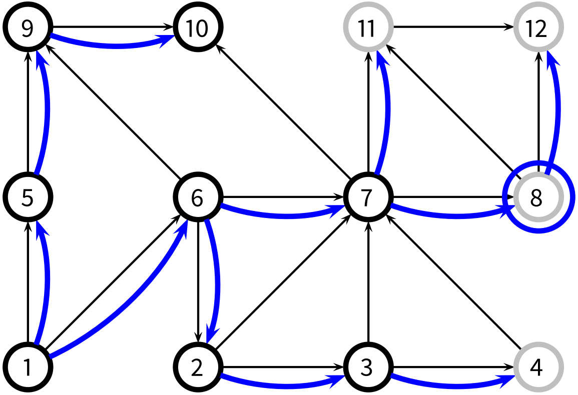


$u = 8$

$Q = \{11, 4\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

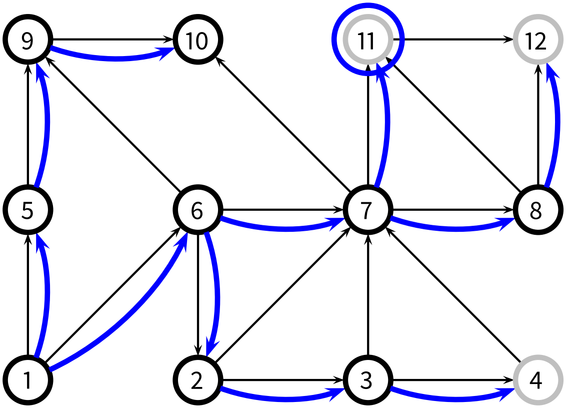


$u = 8$

$Q = \{11, 4, 12\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

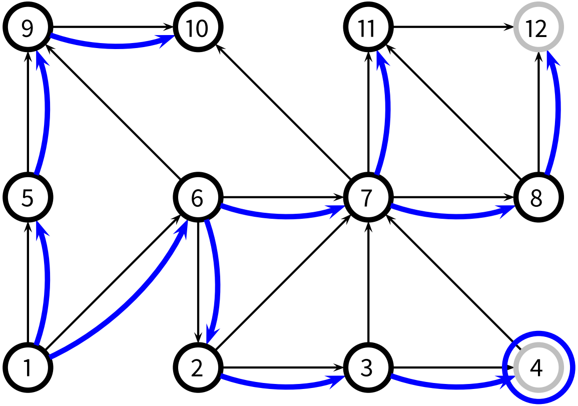


$u = 11$

$Q = \{4, 12\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

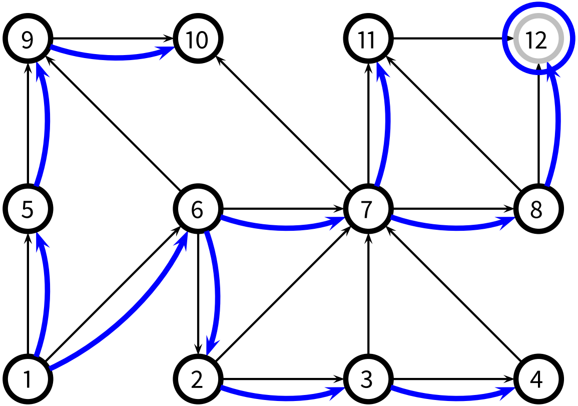


$u = 4$

$Q = \{12\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

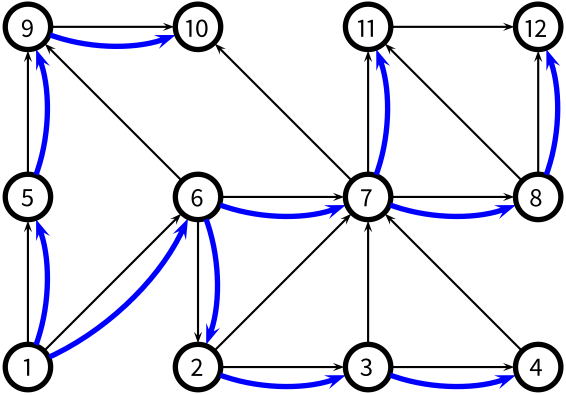


$u = 12$

$Q = \emptyset$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



Complexity of BFS

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
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    d[u] = ∞
    π[u] = NIL
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π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
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    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

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    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
            if color[v] == WHITE
                color[v] = GRAY
                d[v] = d[u] + 1
                π[v] = u
                ENQUEUE(Q, v)
        color[u] = BLACK
```

- We enqueue a vertex only if it is white, and we immediately color it gray; thus, we enqueue every vertex *at most once*

Complexity of BFS

```
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for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
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    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

- We enqueue a vertex only if it is white, and we immediately color it gray; thus, we enqueue every vertex *at most once*
- This means that we also dequeue every vertex *at most once*
- So, the (dequeue) while loop executes $O(|V|)$ times

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color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
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while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
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            ENQUEUE(Q, v)
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- This means that we also dequeue every vertex *at most once*
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- The inner loop: For each vertex u, the inner loop executes for no more than out-deg(u) times, for a total of $\sum_{u \in V}$ out-deg$(u) =$

Complexity of BFS

```
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        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

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Complexity of BFS

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

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- This means that we also dequeue every vertex *at most once*
- So, the (dequeue) while loop executes $O(|V|)$ times
- The inner loop: For each vertex u, the inner loop executes for no more than out-deg(u) times, for a total of $\sum_{u \in V}$ out-deg(u) $= |E|$ times
- So, $O(|V| + |E|)$ (because $|E|$ may be less than $|V|$)

BFS gives the shortest-path distance

Claim

Let $\delta(s, v)$ be the minimum numbers of edges of any path from s to v. Then, we claim that $d[v] = \delta(s, v)$.

BFS gives the shortest-path distance

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Let $\delta(s, v)$ be the minimum numbers of edges of any path from s to v. Then, we claim that $d[v] = \delta(s, v)$.

Reason:

- First, we observe that, before a vertex with distance $k + 1$ is visited, all vertices with distance $\leq k$ have been visited
  - At the beginning, there is only one vertex s with distance ≤ 0, and s is the first vertex visited. When visiting s, we discover all vertices with distance ≤ 1
  - When visiting vertices with distance 1, we discover all vertices with distance ≤ 2
  - ... (The predecessor of every vertex with distance k must be a vertex with distance $k - 1$)

BFS gives the shortest-path distance

Claim

Let $\delta(s, v)$ be the minimum numbers of edges of any path from s to v. Then, we claim that $d[v] = \delta(s, v)$.

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  - At the beginning, there is only one vertex s with distance ≤ 0, and s is the first vertex visited. When visiting s, we discover all vertices with distance ≤ 1
  - When visiting vertices with distance 1, we discover all vertices with distance ≤ 2
  - ... (The predecessor of every vertex with distance k must be a vertex with distance $k - 1$)
- For contradiction, suppose the claim is not true, and consider the first vertex v visited by BFS s.t. $\delta(s, v) \neq d[v]$
- Assume v is discovered when visiting u. We have $d[v] = d[u] + 1$.
- Since $\delta(s, v) \leq d[v]$, we have that $\delta(s, v) < d[v]$.

BFS gives the shortest-path distance

Claim

Let $\delta(s, v)$ be the minimum numbers of edges of any path from s to v. Then, we claim that $d[v] = \delta(s, v)$.

Reason:

- First, we observe that, before a vertex with distance $k + 1$ is visited, all vertices with distance $\leq k$ have been visited
  - At the beginning, there is only one vertex s with distance ≤ 0, and s is the first vertex visited. When visiting s, we discover all vertices with distance ≤ 1
  - When visiting vertices with distance 1, we discover all vertices with distance ≤ 2
  - ... (The predecessor of every vertex with distance k must be a vertex with distance $k - 1$)
- For contradiction, suppose the claim is not true, and consider the first vertex v visited by BFS s.t. $\delta(s, v) \neq d[v]$
- Assume v is discovered when visiting u. We have $d[v] = d[u] + 1$.
- Since $\delta(s, v) \leq d[v]$, we have that $\delta(s, v) < d[v]$.
- Let w be the predecessor of v on a shortest path from s to v. We have $\delta(s, w) = \delta(s, v) - 1$. So $\delta(s, w) < d[v] - 1 = d[u] = \delta(s, u)$.
- This means that w must be visited before u, and when we visit w, we must have marked v as gray. This contradicts the fact that when we visit u, the color of v is still white.

Why the edges $(\pi[v], v)$ form a tree?

Here we prove that the underlying undirected graph is an (undirected) tree

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- The number of tree edges: $n_0 - 1$
- The underlying undirected graph formed by these vertices and edges is definitely connected (because we are only attaching an edge to the partial graph we are building)
- Previous fact: A connected, undirected graph with n vertices and $n-1$ edges is a tree

Dijkstra's Algorithm

- Assumes all edges have ***non-negative*** weights
- A ***greedy*** algorithm

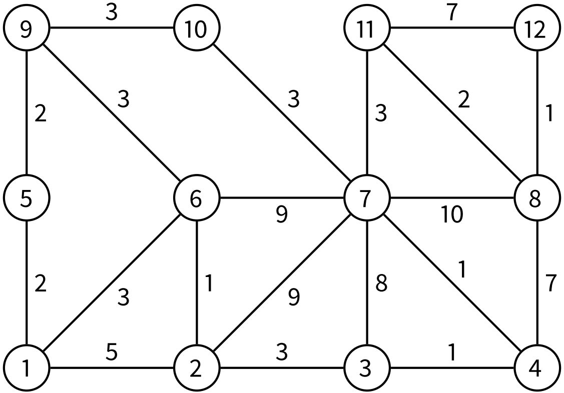
Dijkstra's Algorithm

```
DIJKSTRA(G, s, w)
N = ∅
for each vertex v ∈ V(G)
        D[v] = ∞
        P[v] = NIL
D[s] = 0
while N ≠ V(G)
        find u ∉ N such that D[u] is minimal
        N = N ∪ {u}
        for all v ∈ Adj(u) \ N
                if D[u] + w(u, v) < D[v]
                        D[v] = D[u] + w(u, v)
                        P[v] = u
```

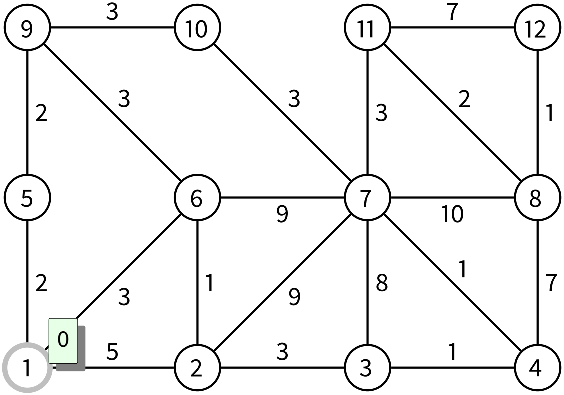
(all algorithms for S-S have the same ***initialization*** and ***relaxation*** process)

- Grows a shortest-path tree from s
  - N: vertices in the (partial) shortest path tree
  - $P[v]$: parent of v in the (partial) shortest path tree (also the vertex preceding v on the shortest path from s)
- Maintains an 'estimate' of the distance to v
  - $D[v]$: weight of the shortest path from s to v where all edges other than the last is from the partial tree
- In each step, makes a greedy choice by adding to the tree a vertex u with minimum value of D
- After adding u to the tree, updates $D[v]$ for the neighbors of u outside the tree if needed (*relaxation*)
- Stops when the tree spans the graph

Example



Example



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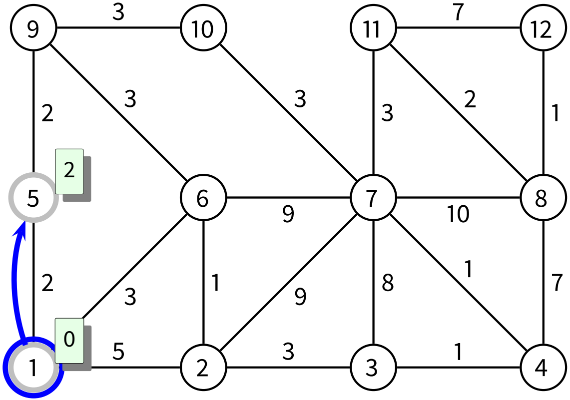

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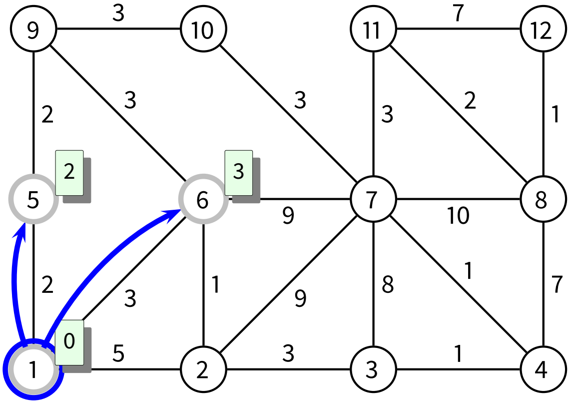

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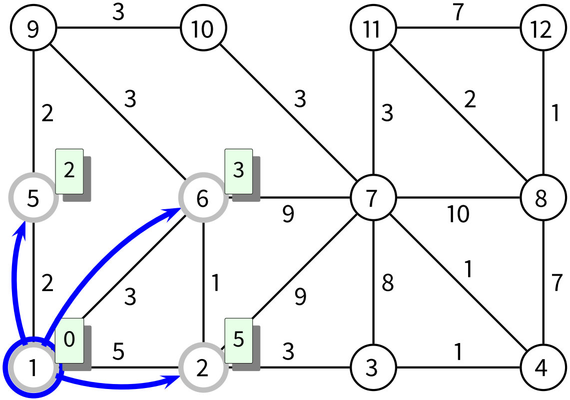

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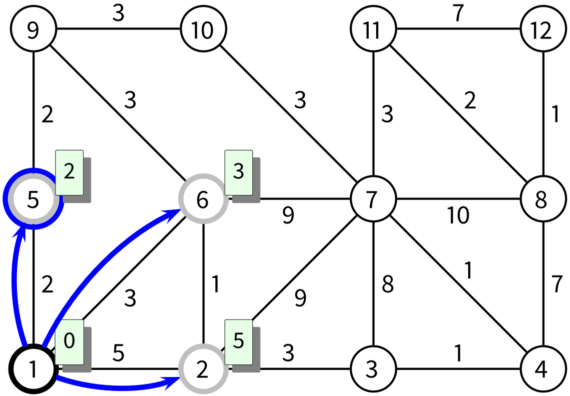

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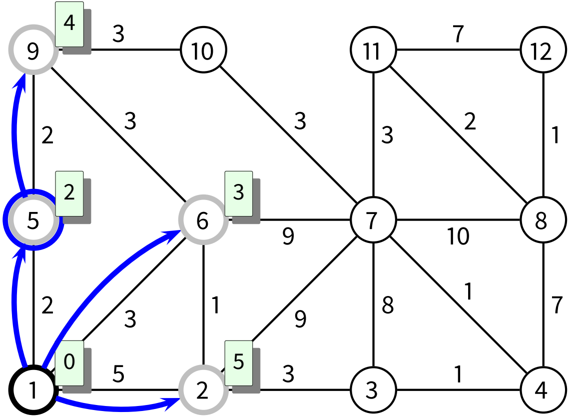

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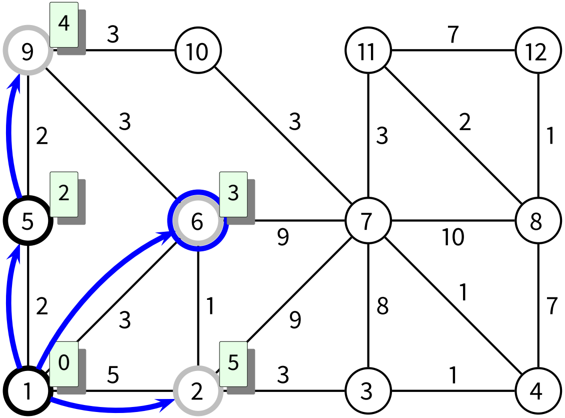

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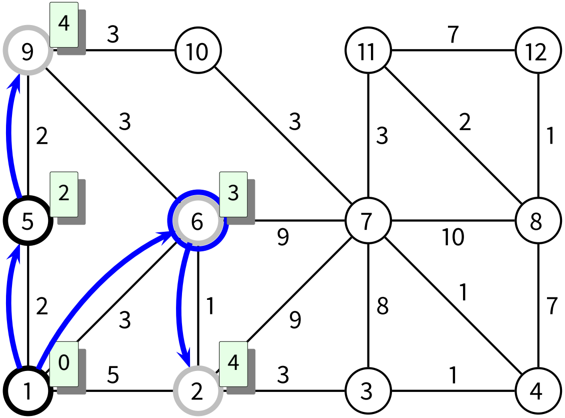

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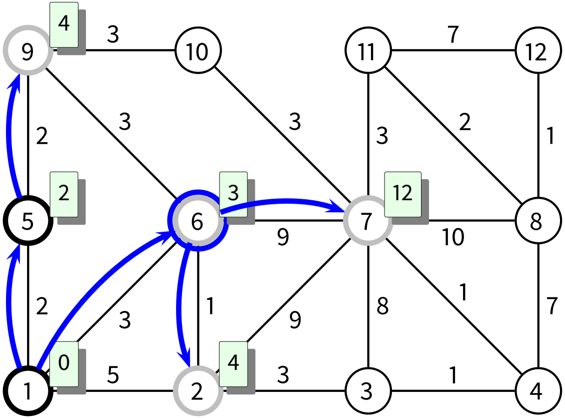

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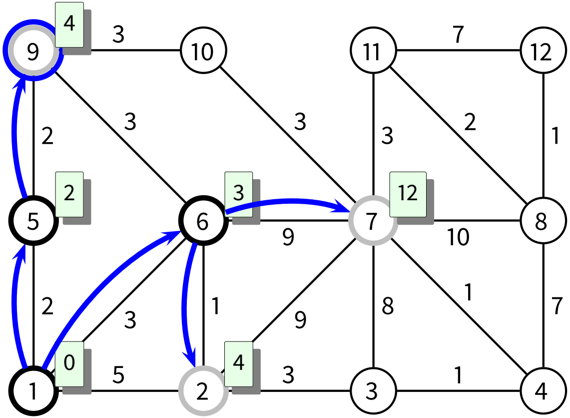

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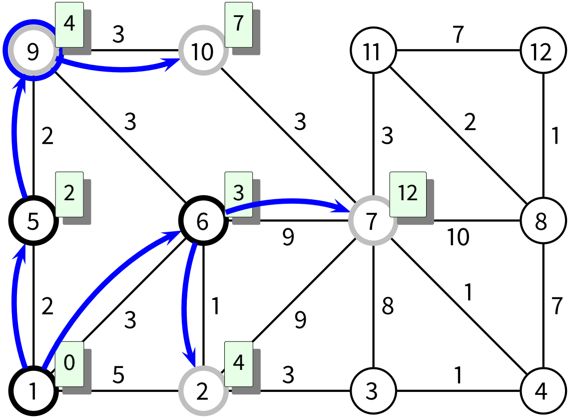

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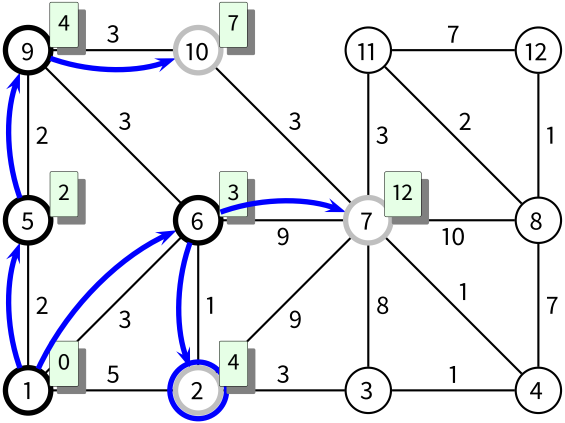

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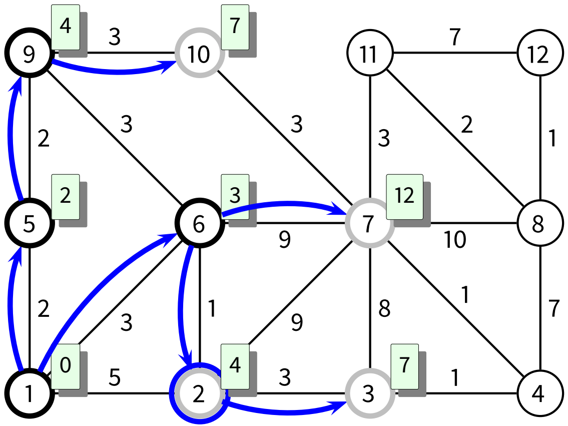

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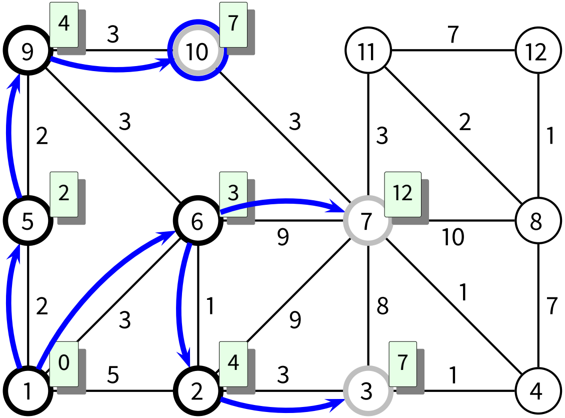

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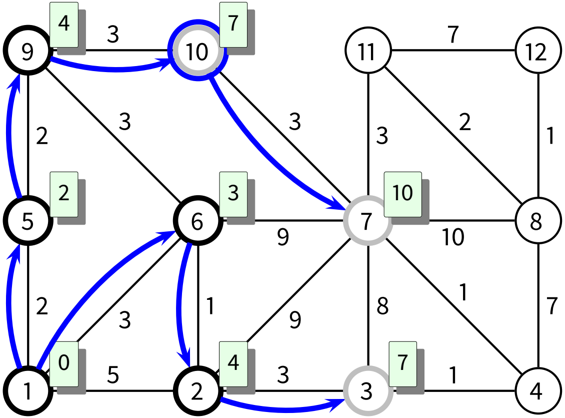

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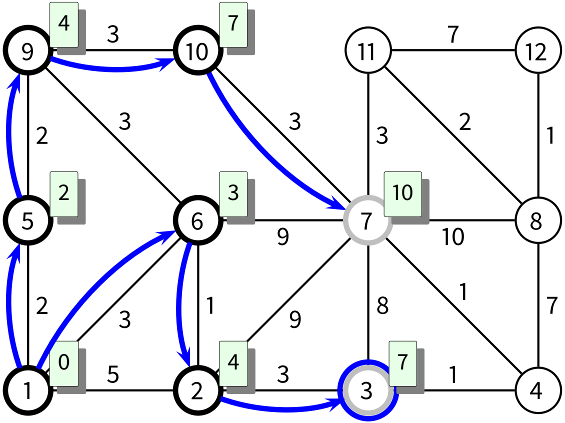

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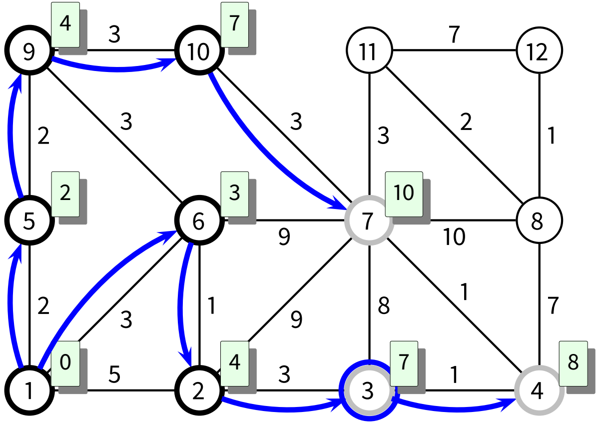

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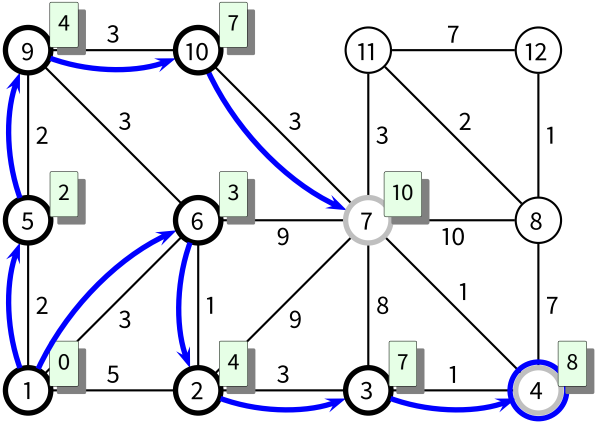

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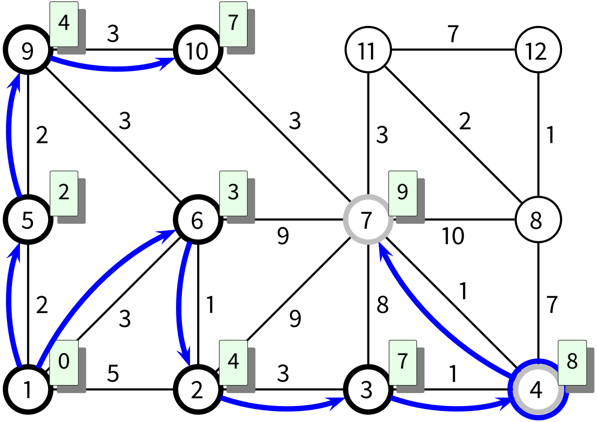

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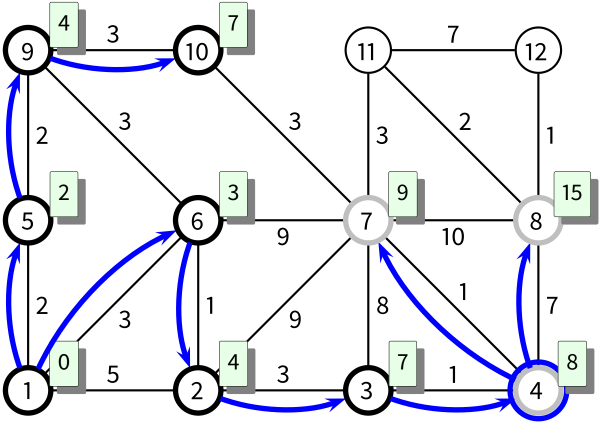

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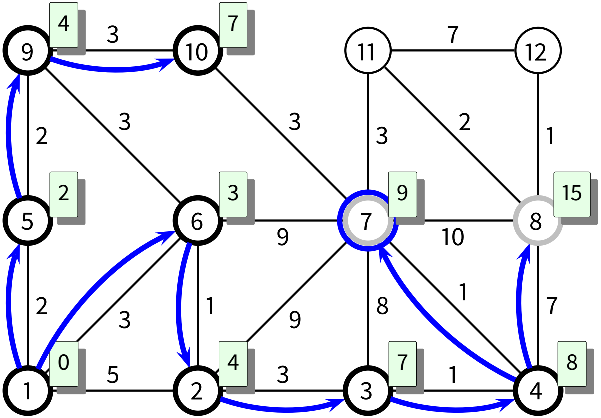

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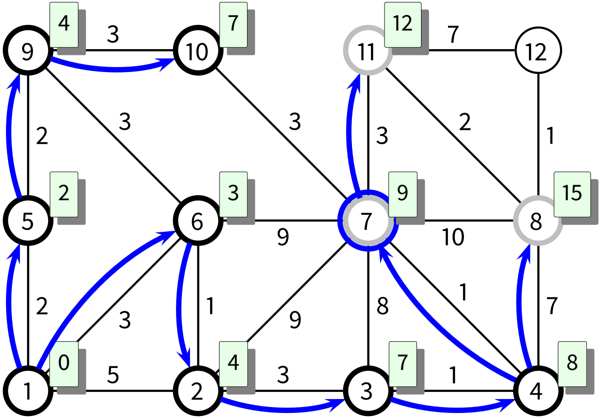

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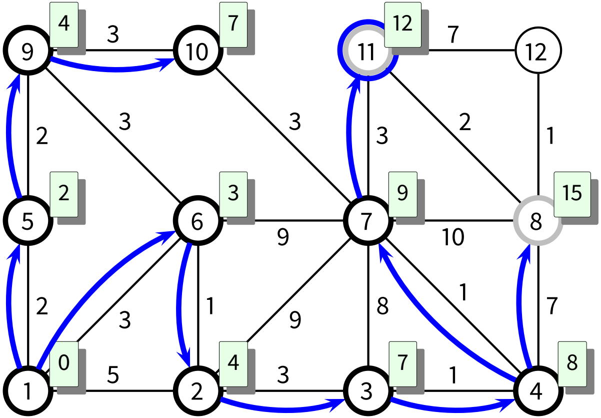

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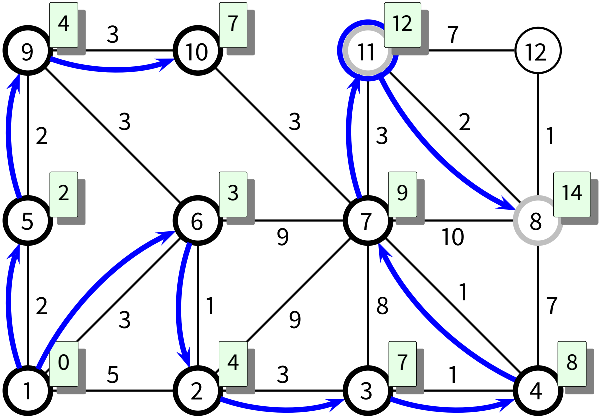

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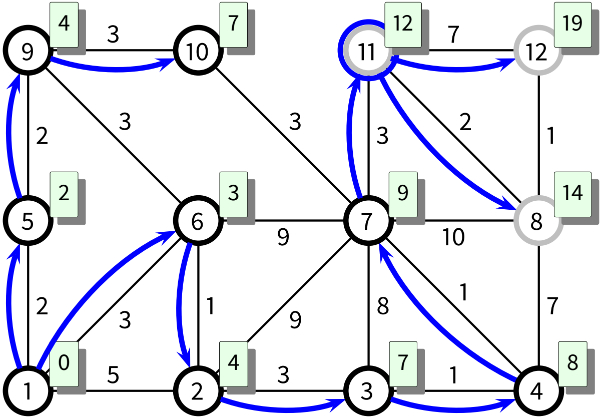

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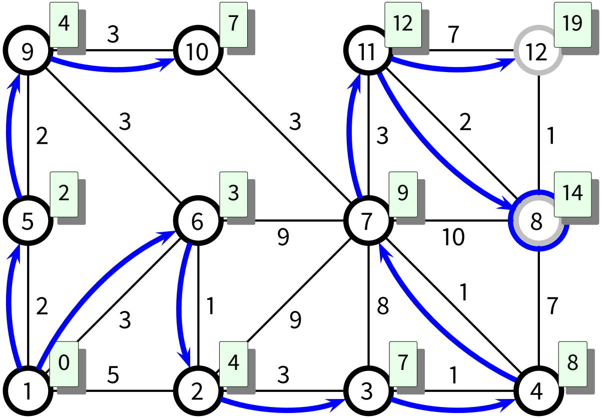

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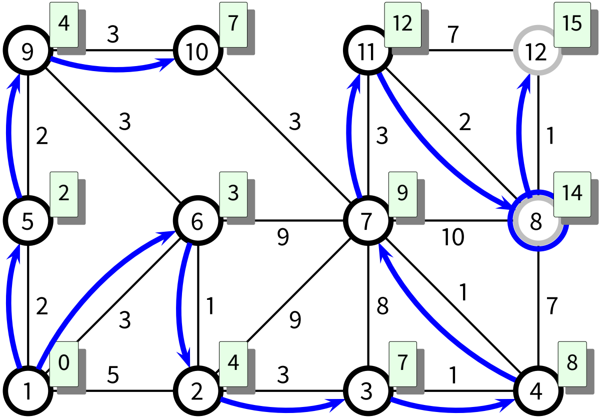

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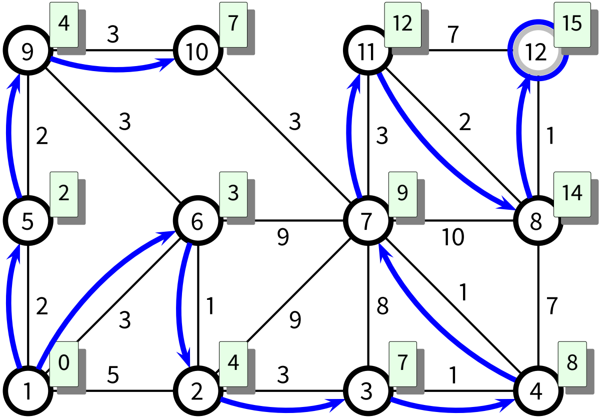
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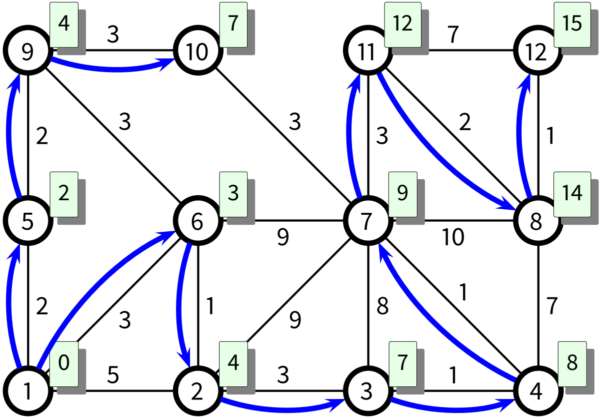

Dijkstra's Algorithm: Complexity

```
DIJKSTRA(G, s, w)
1   N = ∅
2   for each vertex v ∈ V(G)
3       D[v] = ∞
4       P[v] = NIL
5   D[s] = 0
6   while N ≠ V(G)
7       find u ∉ N such that D[u] is minimal
8       N = N ∪ {u}
9       for all v ∈ Adj(u) \ N
10          if D[u] + w(u, v) < D[v]
11              D[v] = D[u] + w(u, v)
12              P[v] = u
```

- Use a heap H for storing D
  - Line 7: EXTRACT-MIN from H
  - Line 11: UPDATE-KEY in H
- Time complexity: $O(|E| \log |V|)$

Dijkstra's Algorithm: Correctness

Some facts:

- If there is no path from s to a v, then $D[v] = \infty$ at all time in the algorithm:
  - Observe the following: whenever $D[v] < \infty$, it always corresponds to a path from s to v
- $\delta(s, v) \leq D[v]$ for all v
  - Again, whenever $D[v] < \infty$, it always corresponds to a path from s to v

Loop variant:

At the start of each iteration of the **while** loop, $D[v] = \delta(s, v)$ for each $v \in N$

Dijkstra's Algorithm: Correctness

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At the start of each iteration of the **while** loop, $D[v] = \delta(s, v)$ for each $v \in N$

Proof:

- Initially, $N = \emptyset$ and this is trivially true
- We only need to show that the invariant is true for all the following iterations

Loop variant:

At the start of each iteration of the **while** loop, $D[v] = \delta(s, v)$ for each $v \in N$

Proof:

- Initially, $N = \varnothing$ and this is trivially true
- We only need to show that the invariant is true for all the following iterations
- For contradiction, let u be the first vertex for which $D[u] \neq \delta(s, u)$ when it is added to N
  - We must have $u \neq s$ because s is the first vertex added to N and $\delta(s, s) = D[s] = 0$; we also have that $N \neq \varnothing$ before u is added to N

Loop variant:

At the start of each iteration of the **while** loop, $D[v] = \delta(s, v)$ for each $v \in N$

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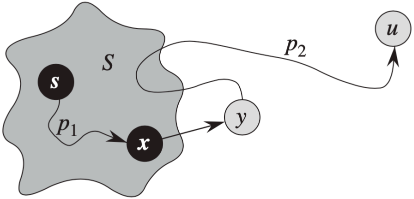
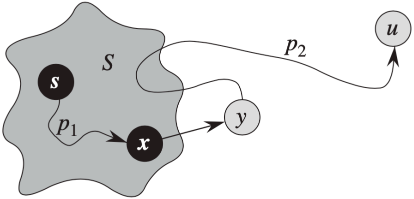
- Since $D[u] \neq \delta(s, u)$, there must be a path from s to u, because otherwise $D[u] = \infty$ for always (previous facts) and $D[u] = \delta(s, u) = \infty$
- Let p be the shortest path from s to u
- Consider the N before adding u: Since the start of p is $s \in N$ and the end of p is $u \notin N$, we can let y be the first vertex along p such that $y \notin N$, and let x be predecessor of y along p ($x \in N$).

Dijkstra's Algorithm: Correctness



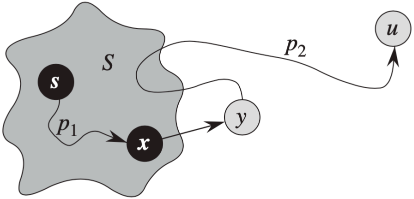
■ (Figure from CLRS)

Dijkstra's Algorithm: Correctness



- (Figure from CLRS)
- Because u is the first vertex added to N for which $D[u] \neq \delta(s, u)$, we have $D[x] = \delta(s, x)$ when x was added

Dijkstra's Algorithm: Correctness

- 

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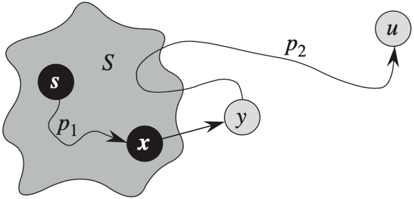
Dijkstra's Algorithm: Correctness

- 

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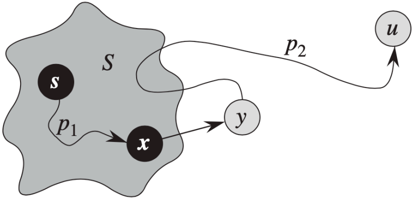
$$D[y] = \delta(s, y) \leq \delta(s, u) \leq D[u]$$

Dijkstra's Algorithm: Correctness

- 

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- From the path p, we know that $\delta(s,y) \leq \delta(s,u)$. So

$$D[y] = \delta(s,y) \leq \delta(s,u) \leq D[u]$$

- Because both u and y were not in N when u was chosen in Line 7, we have $D[u] \leq D[y]$. So the above become equalities

$$D[y] = \delta(s,y) = \delta(s,u) = D[u]$$

A contradiction!

Bellman-Ford algorithm

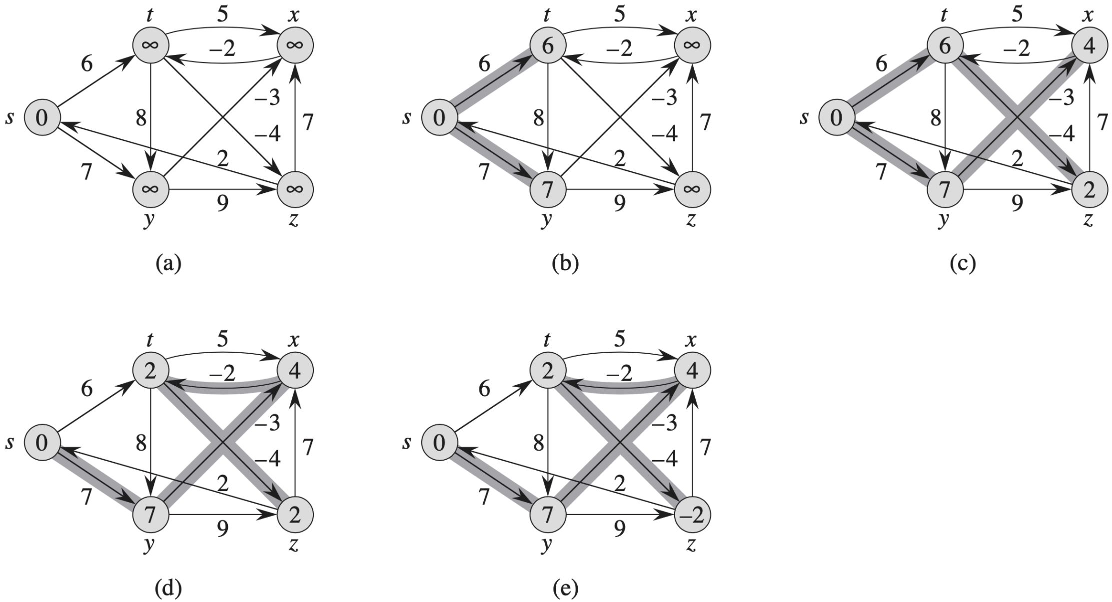
- The ***most general-purpose*** algorithm for computing single-source shortest paths: allows ***negative weights*** on edges, and works for ***any*** graphs
- Returns a ***boolean*** value indicating whether or not there is a negative-weight cycle that is reachable from the source
  - If there is such a cycle, the algorithm indicates that ***no solution*** exists.
  - If there is no such cycle, the algorithm produces the shortest paths and their weights.
- The idea of the algorithm is simple, after the initialization (common to all S-S shortest path algorithms), it has $|V| - 1$ ***rounds***, where each round ***relaxes all the edges***

Bellman-Ford algorithm

```
BELLMAN-FORD(G = (V, E), s, w)
for each vertex v ∈ V
    D[v] = ∞
    P[v] = NIL
D[s] = 0
for i = 1, . . . , |V| − 1
    for each edge (u, v) ∈ E
        RELAX(u, v)
for each edge (u, v) ∈ E
    if D[u] + w(u, v) < D[v]
        return FALSE
return TRUE
```

```
RELAX(u, v)
if D[u] + w(u, v) < D[v]
    D[v] = D[u] + w(u, v)
    P[v] = u
```

Time complexity: $O(|V| \times |E|)$



(Example from CLRS)

Proposition 1

We always have $D[v] \geq \delta(s, v)$ in the algorithm. Furthermore, after $D[v] = \delta(s, v)$, $D[v]$ does not change no matter what relaxations we perform in the algorithm.

Proof:

- We have seen the argument for $D[v] \geq \delta(s, v)$ before, i.e., $D[v]$ always corresponds to the weight of an actual path from s to v (or else $D[v] = \infty$), and so it cannot be less than the optimal one, $\delta(s, v)$.
- For the second part, notice that a relaxation always decreases $D[x]$ for a vertex x. If we already have $D[v] = \delta(s, v)$, then $D[v]$ cannot be further decreased because $D[v] \geq \delta(s, v)$.

Path-relaxation property

Let $p = \langle v_0 = s, v_1, \ldots, v_k \rangle$ be a shortest path from s to v_k. After relaxing the edges in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, we have $D[v_k] = \delta(s, v_k)$. This property holds even if there are relaxations of other edges intermixed with relaxations of the edges on p.

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Proof:

- We prove by induction that, for each i, before relaxing (v_i, v_{i+1}), we have $D[v_i] = \delta(s, v_i)$, and after the relaxation, we have $D[v_{i+1}] = \delta(s, v_{i+1})$.

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- Unlike other proofs by inductions, we prove the **induction step** first.
- Assume for $i - 1$, the claim is true. We have that after relaxing (v_{i-1}, v_i), $D[v_i] = \delta(s, v_i)$ (inductive assumption).
- After that, no matter what relaxations we perform, we always have $D[v_i] = \delta(s, v_i)$ (Proposition 1).

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- Then, before relaxing (v_i, v_{i+1}), we have $D[v_i] = \delta(s, v_i)$ (first part is true)

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- When relaxing (v_i, v_{i+1}), if $D[v_{i+1}] = \delta(s, v_{i+1})$ already, then we have nothing to prove.

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- When relaxing (v_i, v_{i+1}), if $D[v_{i+1}] = \delta(s, v_{i+1})$ already, then we have nothing to prove.
- If $D[v_{i+1}] > \delta(s, v_{i+1})$, then $D[v_{i+1}] > \delta(s, v_{i+1}) = \delta(s, v_i) + w(v_i, v_{i+1}) = D[v_i] + w(v_i, v_{i+1})$, so $D[v_{i+1}]$ must be updated to $D[v_i] + w(v_i, v_{i+1}) = \delta(s, v_{i+1})$ by the relaxation.

Path-relaxation property

Let $p = \langle v_0 = s, v_1, \ldots, v_k \rangle$ be a shortest path from s to v_k. After relaxing the edges in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, we have $d[v_k] = \delta(s, v_k)$. This property holds even if there are relaxations of other edges intermixed with relaxations of the edges on p.

Proof:

- For the **base case**, we first have that $D[v_0] = 0$ always for $v_0 = s$.
- The verification for the rest of the base case is the same as that for the induction step.

Correctness of Bellman-Ford

The two lemmas combined indicate that Bellman-Ford is correct:

Lemma 1

If G contains no negative-weight cycles reachable from s, then the algorithm returns TRUE, and we have $D[v] = \delta(s, v)$ for every vertex $v \in V$.

Lemma 2

If G contains a negative-weight cycle reachable from s, then the algorithm returns FALSE.

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Proof:

- If a vertex v is reachable from s, then let $p = \langle u_0 = s, u_1, \ldots, u_k = v \rangle$ be a shortest path from s to v.
- Since p contains no cycle, then the number of vertices on the path is $k + 1 \leq |V|$ (i.e, $k \leq |V| - 1$).
- So we have
  - The first round ($i = 1$) relaxes (u_0, u_1)
  - The second round ($i = 2$) relaxes (u_1, u_2)
  - ...
  - The k-th round ($i = k \leq |V| - 1$) relaxes (u_{k-1}, u_k)

  After the relaxations, $D[v] = \delta(s, v)$ (by Path-relaxation property)

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  After the relaxations, $D[v] = \delta(s, v)$ (by Path-relaxation property)
- If a vertex v is not reachable from s, then we have that $D[v] = \infty = \delta(s, v)$ always.

Lemma 1

If G contains no negative-weight cycles reachable from s, then the algorithm returns TRUE, and we have $D[v] = \delta(s, v)$ for every vertex $v \in V$.

Proof:

- We still need to prove that the algorithm returns TRUE.

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If G contains no negative-weight cycles reachable from s, then the algorithm returns TRUE, and we have $D[v] = \delta(s, v)$ for every vertex $v \in V$.

Proof:

- We still need to prove that the algorithm returns TRUE.
- For each edge (u, v), we have

$$\begin{aligned} D[v] &= \delta(s, v) && (1) \\ &\leq \delta(s, u) + w(u, v) && (2) \\ &= D[u] + w(u, v) && (3) \end{aligned}$$

(1)-(2) follows from 'triangle inequality'

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If G contains a negative-weight cycle reachable from s, then the algorithm returns FALSE.

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Proof:

- Let $c = \langle v_0, v_1, \ldots, v_k \rangle$ be a negative-weight cycle reachable from s where $v_0 = v_k$.
- We have $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$.

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- For contradiction, assume the algorithm returns TRUE.
- Then, $D[v_i] \leq D[v_{i-1}] + w(v_{i-1}, v_i)$, for $i = 1, \ldots, k$

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- Then, $D[v_i] \leq D[v_{i-1}] + w(v_{i-1}, v_i)$, for $i = 1, \ldots, k$
- Summing all the above inequalities:

$$\begin{aligned}
\sum_{i=1}^{k} D[v_i] &\leq \sum_{i=1}^{k} (D[v_{i-1}] + w(v_{i-1}, v_i)) \\
&= \sum_{i=1}^{k} D[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)
\end{aligned}$$

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$$\begin{aligned}\sum_{i=1}^{k} D[v_i] &\leq \sum_{i=1}^{k} (D[v_{i-1}] + w(v_{i-1}, v_i)) \\ &= \sum_{i=1}^{k} D[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)\end{aligned}$$

- Since $v_0 = v_k$, we have $\sum_{i=1}^{k} D[v_i] = \sum_{i=1}^{k} D[v_{i-1}]$

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- Then, $D[v_i] \leq D[v_{i-1}] + w(v_{i-1}, v_i)$, for $i = 1, \ldots, k$
- Summing all the above inequalities:

$$\begin{aligned}\sum_{i=1}^{k} D[v_i] &\leq \sum_{i=1}^{k} (D[v_{i-1}] + w(v_{i-1}, v_i)) \\ &= \sum_{i=1}^{k} D[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)\end{aligned}$$

- Since $v_0 = v_k$, we have $\sum_{i=1}^{k} D[v_i] = \sum_{i=1}^{k} D[v_{i-1}]$
- So the inequality becomes $0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i)$ (***contradiction***)

Single-source shortest paths in DAG

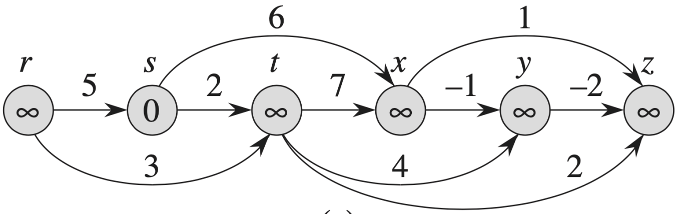
- Assumes the graph is a DAG (directed acyclic graph)
- Edges can have *negative* weights
  - Since we are dealing with DAG, no (negative-weight) cycles can exist
- Finding the shortest-path distance for vertices based on the ***order of topological sort***

Single-source shortest paths in DAG

```
DAG-SHORTEST-PATHS(G, s, w)
topologically sort vertices in G
for each vertex v ∈ V(G)
        D[v] = ∞
        P[v] = NIL
D[s] = 0
for each vertex u ∈ G, in topologically sorted order
        for all v ∈ Adj(u)
                RELAX(u, v)
```

Time complexity: $O(|V| + |E|)$

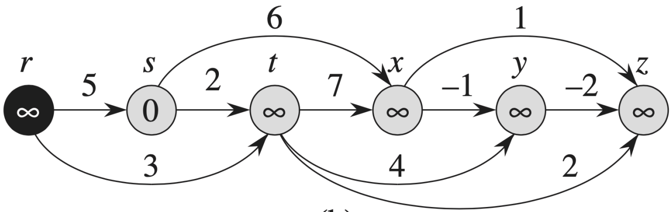
Single-source shortest paths in DAG



(a)

(Example from CLRS)

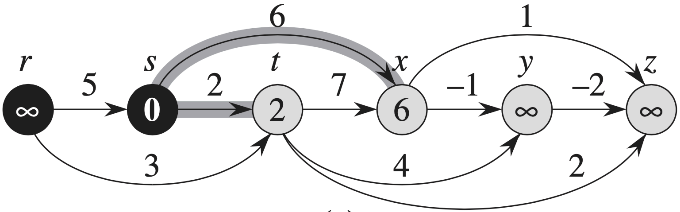
Single-source shortest paths in DAG



(b)

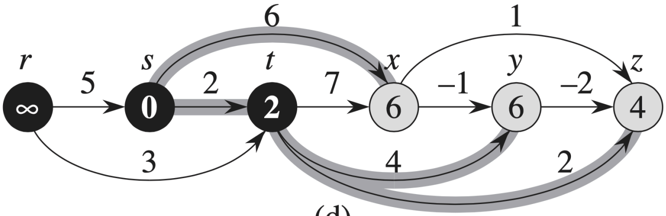
(Example from CLRS)

Single-source shortest paths in DAG



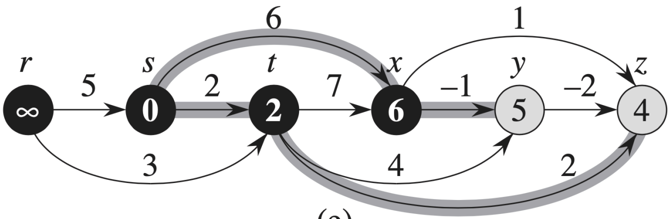
(c)

(Example from CLRS)



(d)

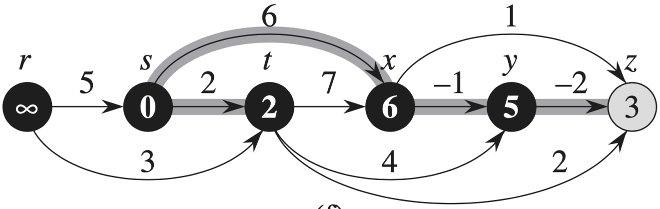
(Example from CLRS)



(e)

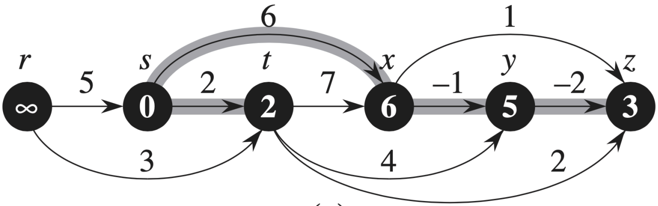
(Example from CLRS)

Single-source shortest paths in DAG



(f)

(Example from CLRS)



(g)

(Example from CLRS)

Proof of correctness

- Hint: The single-source shortest paths algorithm for DAG can be viewed as a ‘smarter’ way of doing Bellman-Ford, and therefore you can adjust the justification for Bellman-Ford to show the correctness of the DAG-algorithm.