Divide and Conquer

Tao Hou

The Divide-and-Conquer Paradigm

- **Divide phase:** Divide the problem into subproblems
- **Conquer phase:** Conquer/solve the subproblems (recursively)
- **Combine phase:** Combine the solutions to the subproblems into a solution for the whole problem

Example: Merge Sort (Review)

- **Divide phase:** Divide the array into two halves from the middle
- **Conquer Phase:** Sort each half recursively
- **Combine phase:** Merge the two sorted halves

Merge Sort

```
MERGESORT(A)
if length(A) == 1
        return A
m = ⌊length(A)/2⌋
A_L = MERGESORT(A[1 .. m])
A_R = MERGESORT(A[m + 1 .. length(A)])
return MERGE(A_L, A_R)
```

- What the MERGE routine does: given two sorted arrays, return a single sorted array containing all elements of the given two arrays
- The MERGE routine runs in $O(n)$ time where n is the size of the larger given array

MERGE Algorithm

```
MERGE(A, B)
i, j = 1
X = ∅
while i ≤ length(A) and j ≤ length(B)
    if A[i] ≤ B[j]
        X = X ∘ A[i]            // appends A[i] to X
        i = i + 1
    else
        X = X ∘ B[j]
        j = j + 1
while i ≤ length(A)
    X = X ∘ A[i]
    i = i + 1
while j ≤ length(B)
    X = X ∘ B[j]
    j = j + 1
return X
```

Run-Time Analysis of Merge Sort

Input Size: n

$$T(n) = \begin{cases} C_1 & \text{if n=1} \\ 2T(n/2) + n * C_2 & \text{otherwise} \end{cases}$$

Q: How to solve it?

The Master Theorem

Let $a \geq 1$, $b > 1$, $f(n) = O(n^d)$ where $d \geq 0$, and $c = \log_b a$

$$T(n) = \begin{cases} O(1) & \text{if } n = O(1) \\ aT(n/b) + f(n) & \text{otherwise} \end{cases}$$

1. $c < d$: $T(n) = \Theta(f(n)) = \Theta(n^d)$
2. $c > d$: $T(n) = \Theta(n^c)$
3. $c = d$: $T(n) = \Theta(n^c \log n)$

The Master Theorem

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2. $c > d$: $T(n) = \Theta(n^c)$
3. $c = d$: $T(n) = \Theta(n^c \log n)$

Remark: For case 1, $f(n)$ must also satisfy a *regularity condition* which states that there is some $C < 1$ such that $a \cdot f(n/b) \leq C \cdot f(n)$ for sufficiently large n. This regularity condition is almost always true and we will not worry about it.

Run-Time Analysis of Merge Sort using Master Theorem

$$T(n) = \begin{cases} C_1 & \text{if n=1} \\ 2T(n/2) + C_2 \cdot n & \text{otherwise} \end{cases}$$

Applying the Master Theorem with $a = 2$, $b = 2$, and $d = 1$, we get $c = \log_2 2 = d$ and $T(n) = \Theta(n \log n)$

Master Theorem: Additional Examples

Example 1

$$T(n) = T(n/2) + 5n$$

Applying the Master Theorem with $a = 1$, $b = 2$, $d = 1$, we get $c = 0 < d$ and hence $T(n) = \Theta(n)$

Master Theorem: Additional Examples

Example 1

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Applying the Master Theorem with $a = 1$, $b = 2$, $d = 1$, we get $c = 0 < d$ and hence $T(n) = \Theta(n)$

Example 2

$$T(n) = 4T(n/2) + 2n$$

Applying the Master Theorem with $a = 4$, $b = 2$, $d = 1$, we get $c = 2 > d$ and hence $T(n) = \Theta(n^2)$

Examples: Using the Master Theorem

Example 3

$$T(n) = T(n-5) + n$$

- The Master Theorem does not apply here.
- The *iteration method* (briefly reviewed next) can be used to solve this equation

Run-Time Analysis of Merge Sort (Iteration Method)

We can also solve $T(n)$ using the *Iteration Method* (aka. keep on expanding the formula by applying $T(n)$ to itself, until reaching the base case):

$$
\begin{aligned}
(1) : T(n) &= 2T(n/2) + C_2 n \\
(2) : T(n) &= 2^2 T(n/2^2) + 2C_2 n \\
(3) : T(n) &= 2^3 T(n/2^3) + 3C_2 n \\
&\vdots \\
(i) : T(n) &= 2^i T(n/2^i) + i \cdot C_2 n
\end{aligned}
$$

We stop iterating when $n/2^i = 1$

Setting $n/2^i = 1$ gives a number of iterations $i = \log n$

Plugging the value of $i = \log n$ gives:

$T(n) = 2^i T(n/2^i) + i \cdot C_2 n = 2^{\log n} C_1 + n \cdot \log n = nC_1 + \log n \cdot C_2 \cdot n = \Theta(n \log n)$

Quicksort (Review)

- ***Divide:*** Partition A into $A[1 \dots q-1]$ and $A[q+1 \dots n]$ such that

$$A[1], \dots, A[q-1] \le A[q] \le A[q+1], \dots, A[n]$$

  - The partition is done by a PARTITION procedure which may change the positions of elements
  - q is returned from the partition procedure and in general we don't have any control over q

- ***Conquer:*** Sort $A[1 \dots q-1]$ and $A[q+1 \dots n]$ recursively

- ***Combine:*** Nothing to do here

```
QUICKSORT(A, begin, end)
if begin < end
    q = PARTITION(A, begin, end)
    QUICKSORT(A, begin, q − 1)
    QUICKSORT(A, q + 1, end)
```

Partition

```
PARTITION(A, begin, end)
q = begin
v = A[end]
for i = begin to end − 1
    if A[i] < v
        swap A[i] and A[q]
        q = q + 1
swap A[q] and A[end]
return q
```

- Runs in $\Theta(n)$ time
- Further remarks:
  - Assume a *pivot* (center of the partition) v to be at the end
  - Loop invariant (always **true** at the beginning of each iteration):
    q is a separation of $A[begin \ldots i-1]$ s.t.

$$A[begin], \ldots, A[q-1] < v \text{ and } A[q], \ldots, A[i-1] \geq v$$

Worst-Case Run-Time Analysis of Quick Sort (Review)

Input Size: n
Worst Case: The array partition is very skewed: 0 element on one side, pivot, and the rest on the other side (the pivot is the smallest or largest element)

$$T(n) = \begin{cases} 1 & \text{if n=1} \\ T(n-1) + n & \text{otherwise} \end{cases}$$

We cannot solve $T(n)$ using the master method.

Using the Iteration Method

We solve $T(n)$ by expanding the recursive formula directly:

$$
\begin{aligned}
(1): T(n) &= T(n-1) + n \\
(2): T(n) &= T(n-2) + n - 1 + n \\
(3): T(n) &= T(n-3) + n - 2 + n - 1 + n \\
&\vdots \\
(i): T(n) &= T(n-i) + (n-i+1) + (n-i+2) + \cdots + n
\end{aligned}
$$

We stop expanding when $n - i = 1$
Setting $n - i = 1$ gives $i = n - 1$
Plugging this value of i in the generic form gives
$T(n) = T(1) + 2 + 3 + \cdots + n = 1 + 2 + 3 + \cdots + n = n(n+1)/2 = \Theta(n^2)$

Average-Case Run-Time Analysis of Quick Sort (Advanced)

- Idea: count the number of comparisons
- Rename elements (assumed to be distinct) in A as $z_1 < z_2 < \cdots < z_n$
- Define a random variable X_{ij} as:

$$X_{ij} = \begin{cases} 0 & \text{if } z_i \text{ and } z_j \text{ does not compare} \\ 1 & \text{if } z_i \text{ and } z_j \text{ does compare} \end{cases}$$

- The random variable for the number of comparison is:

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

Average-Case Run-Time Analysis of Quick Sort (Advanced)

- We have

$$E[X] = E\Big[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\Big]$$
$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

- By some analysis (we omit),

$$E[X_{ij}] = \frac{2}{j - i + 1}$$

Average-Case Run-Time Analysis of Quick Sort (Advanced)

- Then

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k} \\
&< \sum_{i=1}^{n-1} \left(2 \sum_{k=1}^{\infty} \frac{1}{k}\right) \qquad \text{(inner sum \textit{harmonic series})} \\
&= \sum_{i=1}^{n-1} O(\log n) = O(n \log n)
\end{aligned}
$$

Selection

Problem

Given an (unsorted) array $A[1 \ldots n]$ of numbers and $k \in \mathbb{N}$, find the k-th smallest number in A

A First Random Solution

(i) **Divide:** Randomly select a pivot from A, partition A into two subarrays L and R s.t. elements in $L \leq$ elements in R

(ii) **Conquer:** If $k \leq |L|$, recurse to find the k-th smallest element in L; otherwise, recurse to find the $(k - |L|)$-th smallest element in R

Random select

RandSelect(A, k)

1. **if** $|A| == 1$ **then return** $A[1]$;
2. $L, R =$ Partition(A);
3. **if** $k \leq |L|$ **then return** RandSelect(L, k);
4. **else return** RandSelect(R, $k - |L|$);

Random solution: Complexity

(Analysis similar to quicksort)

- Best case:

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- Best case: $O(n)$

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- Best case: $O(n)$
- Worst case:

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- Best case: $O(n)$
- Worst case: $O(n^2)$

Random solution: Complexity

(Analysis similar to quicksort)

- Best case: $O(n)$
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- Average case:

Random solution: Complexity

(Analysis similar to quicksort)

- Best case: $O(n)$
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- Average case: $O(n)$

Linear-Time Selection

Can we have a selection algorithm which runs in linear time in the worst case?

- Observe that the previous random selection runs in quadratic time because sometimes the partition can be unbalanced
- Can we try to choose a “good” pivot for the partition each time so that the partitioned arrays are always balanced?

Linear-Time Selection

Can we have a selection algorithm which runs in linear time in the worst case?

- Observe that the previous random selection runs in quadratic time because sometimes the partition can be unbalanced
- Can we try to choose a "good" pivot for the partition each time so that the partitioned arrays are always balanced?
- The answer is that we can

Linear-Time Selection

Solution:

(i) Partition the array into $m = \lceil n/5 \rceil$ subarrays, each consisting of 5 (maybe less) consecutive elements

(ii) Find the median of each of the m arrays by brute force

(iii) Recursively find the median M of the m medians

(iv) Using M as pivot, partition A into two subarrays L and R

(v) If $k \leq |L|$, recurse to find the k-th smallest element in L; otherwise, recurse to find the $(k - |L|)$-th smallest element in R

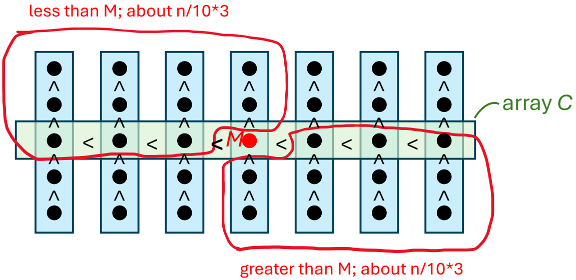
The Selection Algorithm

Select(A, k)

1. **if** $n \leq 25$ **then return** the k-th smallest element in A by brute force;
2. $m = \lceil n/5 \rceil$; create an array $C[1..m]$;
3. **for** $i = 1$ **to** m $C[i] :=$ the median of $A[(5i-4)..(5i)]$;
4. $M = \text{Select}(C,\ m/2)$;
5. Partition A using M as the pivot into L and R, where L contains all elements that are smaller or equal to M and R contains the rest;
6. **if** $k \leq |L|$ **then return** $\text{Select}(L,\ k)$;
7. **else return** $\text{Select}(R,\ k - |L|)$;

Run-Time Analysis of Select

- Take $n = 35$
- For simplicity, assume all elements are distinct
- Order each small array, and then order the 7 small arrays by their medians



Run-Time Analysis of Select

In general:

- Ignore the floors and ceilings
- The number of medians in the array C less than M is: $(1/2) \cdot (n/5) = n/10$
- The number of other elements less than M is at least: $2n/10$
- So, at lease $3n/10$ elements is less than M
- Similarly, at lease $3n/10$ elements is greater than M
- Whether we go to L or R in the algorithm, we drop at least $3n/10$ elements (i.e., keep at most $7n/10$ elements).

Run-Time Analysis of Select

$$T(n) \leq \begin{cases} O(1) & \text{if } n \leq 25 \\ T(7n/10) + T(n/5) + O(n) & \text{otherwise} \end{cases}$$

We cannot solve $T(n)$ using the master method.

Instead, use the *substitution* method:

1. Guess the solution
2. Plug in the guess and prove the equation to be true based on the assumption that the equation is true for sub-cases

Notice: The substitution method is in some sense a proof by *induction*

Run-Time Analysis of Select

- Our induction hypothesis:
  - Suppose that $T(i) \le c \cdot i$ for any $i < n$, where c is a constant
  - Want to prove that $T(n) \le c \cdot n$, which means $T(n) = O(n)$ by definition

Run-Time Analysis of Select

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  - Suppose that $T(i) \leq c \cdot i$ for any $i < n$, where c is a constant
  - Want to prove that $T(n) \leq c \cdot n$, which means $T(n) = O(n)$ by definition
- We have

$$
\begin{aligned}
T(n) &\leq T(7n/10) + T(n/5) + O(n) \\
&\leq c \cdot (7n/10) + c \cdot (n/5) + c'n \\
&= 9cn/10 + c'n = cn \cdot (9/10 + c'/c)
\end{aligned}
$$

Run-Time Analysis of Select

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  - Suppose that $T(i) \leq c \cdot i$ for any $i < n$, where c is a constant
  - Want to prove that $T(n) \leq c \cdot n$, which means $T(n) = O(n)$ by definition
- We have

$$\begin{aligned} T(n) &\leq T(7n/10) + T(n/5) + O(n) \\ &\leq c \cdot (7n/10) + c \cdot (n/5) + c'n \\ &= 9cn/10 + c'n = cn \cdot (9/10 + c'/c) \end{aligned}$$

- So we only need to choose a c s.t. $c'/c + 9/10 \leq 1$, which is $c \geq 10c'$, so that we will have

$$T(n) \leq cn \cdot (9/10 + c'/c) \leq cn$$

The Closest Pair of Points

Problem

Given a set $S = \{p_1, \ldots, p_n\}$ of points in the plane, where $p_i = (x_i, y_i)$, compute a closest-pair of points in S, that is, a pair of distinct points $p_i, p_j \in S$ such that $|p_i p_j| = \min\{|p_r p_s| : \; p_r \neq p_s \in S\}$

Note: we assume the points in S to have *distinct* coordinates; if there are duplicate points in S, this is easy to pre-check and the answer is 0

The Closest-Pair Algorithm: Overview

- **Divide:** Partition the input set S into two sets S_L and S_R of the same size s.t. points in S_L are to the left of points in S_R
- **Conquer:** Recursively find the minimum distances of S_L and S_R
- **Combine:** Find the minimum distance of point pairs where one is from S_L and the other is from S_R; return the minimum of the three minimums

We aim to achieve $O(n)$ time for both the divide and combine phase so that the entire complexity is $O(n \log n)$

Preprocessing Step

- Let X be a list containing the points in S sorted w.r.t. their x-coordinates, and Y a list containing the points in S sorted w.r.t. their y-coordinates. Clearly, X and Y can be obtained in $O(n \log n)$ time (we only do this *once* at the beginning).

So the input to the algorithm, i.e., the set of points, is encoded as a tuple of three arrays (S, X, Y)

Divide Phase

- Partition S into S_L and S_R of equal size s.t. points in S_L are to the *left* of S_R using a central vertical line D
- Let X_L, Y_L each represent the set of points in S_L sorted by x- and y-coordinates respectively; X_R and Y_R are similarly defined for S_R

Divide Phase: Pseudocode

```
m = |X|/2
D = X[m].x
X_L = X[1 ... m]
X_R = X[m + 1 ... |X|]
for i = 1 ... |Y|:
    if Y[i].x ≤ D:
        append Y[i] to Y_L
    else:
        append Y[i] to Y_R
separate S into S_L, S_R similarly
```

Conquer Phase

- Recursively call the algorithm on (S_L, X_L, Y_L) to obtain the min-distance δ_L for S_L, and on (S_R, X_R, Y_R) to obtain the min-distance δ_R for S_R.

Combine Phase

Idea

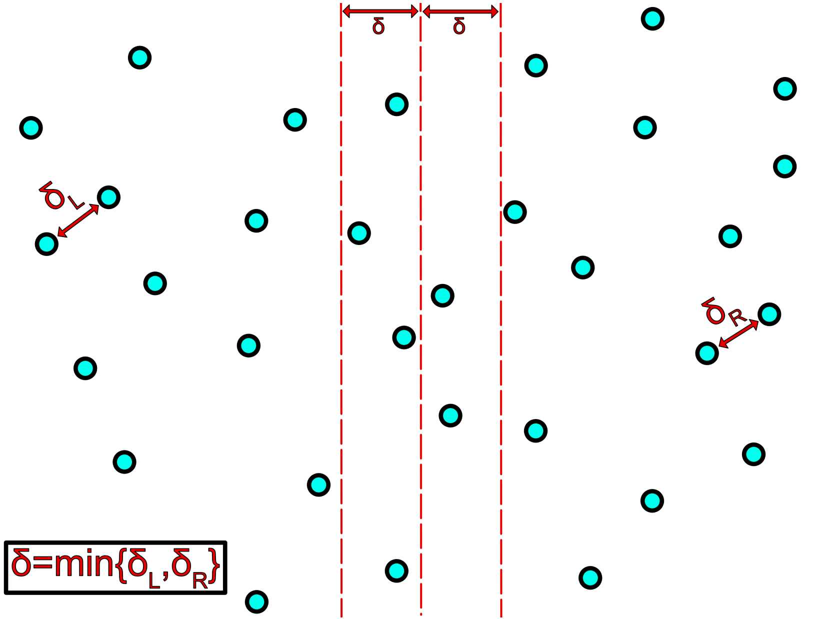
- We have:
  - δ_L: The min dis of pairs in S_L
  - δ_R: The min dis of pairs in S_R
- Aim of combine phase: Compute the min-dis of the pairs where one point is from S_L and the other is from S_R (i.e., pairs of points from different sides)
- Answer: The minimum of above three minimums

Details of Combine Phase

The first observation

- Let $\delta = \min\{\delta_L, \delta_R\}$
- We only need to consider pairs within a 2δ-**wide vertical strip centered around** D

Consider only 2δ-wide vertical strip centered around D



Explanation

- We have computed min-dis of points from the same side, which is δ.
- So, to compute the overall min-dis, we can ignore those point pairs whose distances are greater than δ.
- If two points from different sides are not both from the 2δ-wide vertical strip (at least one point is outside the strip), then their distance is greater than δ, and so we can ignore them.

The Combine Phase

- Let $\delta = \min\{\delta_L, \delta_R\}$
- From Y, create Y_{mid} (also sorted by y-coordinates) which is the set of points within the 2δ-wide vertical strip centered around D

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- Let $\delta = \min\{\delta_L, \delta_R\}$
- From Y, create Y_{mid} (also sorted by y-coordinates) which is the set of points within the 2δ-wide vertical strip centered around D
- Go over Y_{mid}, and for each point p, compute its distance to **at most 7** points in Y_{mid} that follow p, and keep track of the min-distance
- Return the smaller of δ and what we have by scanning Y_{mid}

Combine Phase: Pseudocode

for $i = 1 \ldots |Y|$:
    if $Y[i].x \geq D - \delta$ and $Y[i].x \leq D + \delta$:
        append $Y[i]$ to Y_{mid}
$\bar{\delta} = \infty$
for $i = 1 \ldots |Y_{mid}|$:
    for $j = 1 \ldots 7$:
        if $i + j \leq |Y_{mid}|$ and $\text{dis}(Y_{mid}[i], Y_{mid}[i+j]) < \bar{\delta}$ **then**
            $\bar{\delta} = \text{dis}(Y_{mid}[i], Y_{mid}[i+j])$
return $\min\{\delta, \bar{\delta}\}$

Why only scan 7 points?

- For each point p in Y_{mid}, we only need to consider other points in Y_{mid} whose distances to p is $< \delta$. This means we only need to consider points within a $2\delta \times 2\delta$ square of p.

Why only scan 7 points?

- For each point p in Y_{mid}, we only need to consider other points in Y_{mid} whose distances to p is $< \delta$. This means we only need to consider points within a $2\delta \times 2\delta$ square of p.
- **Key observation**: Each $\delta \times \delta$ square contains at most 4 points
  - This square is totally within the left or right side of the vertical separator D, meaning that points in the square are either all from S_L or all from S_R, so these points are at least δ-distance apart
  - A fact from computational geometry says that such a square cannot fit in more than 4 points

Why only scan 7 points?

- Therefore, each $2\delta \times \delta$ square contains at most 8 points (including p)
- So we only need to scan the 7 points that precede p (ones that are in the upper $2\delta \times \delta$ square) and the 7 points that follow p (ones that are in the lower $2\delta \times \delta$ square) in Y_{mid}.
- Further observation: we only need to scan the 7 points that follow p, and ignore the 7 points that precede p:
  - Suppose there is a point q preceding p in Y_{mid} falling within the upper $2\delta \times \delta$ square for p. Then p also falls in the lower $2\delta \times \delta$ square for q. So we have checked the pair p, q when we scan q.

The Closest-Pair Algorithm

Closest-Pair-Algo

1. **if** $|S| \leq 3$ **return** a closes pair (p_{min}, q_{min}) in S by brute force;
2. using X, compute a vertical line D of equation $x = \ell$ that partitions S into S_L, S_R of equal size such that all points in S_L are on D or to the left of it, and all points in S_R are on D or to the right of it;
3. using X and Y, create the arrays X_L, Y_L and X_R, Y_R;
4. recurse on S_L, X_L, Y_L to compute a closest pair (p_L, q_L); let $\delta_L = |p_L q_L|$;
5. recurse on S_R, X_R, Y_R to compute a closest pair (p_R, q_R); let $\delta_R = |p_R q_R|$;
6. let $\delta = \min\{\delta_L, \delta_R\}$;
7. let S_{mid} be the set of points in S whose x-coordinate satisfies $\ell - \delta \leq x \leq x + \delta$;
8. using Y, compute the list of points in S_{mid} sorted by their y-coordinates;
9. go over Y_{mid} (in the sorted order), and for each point, compute its distance to the next (at most) 7 points in Y_{mid} and keep track of the pair of points (p_{mid}, q_{mid}) of minimum distance;
10. return the closest pair (p_{min}, q_{min}) among, $(p_L, q_L), (p_R, q_R)$, and (p_{mid}, q_{mid});

Run-Time Analysis of Closest-Pair

Let $T(n)$ be the running time of Closest-Pair in the worst case on n points.

- Divide phase takes $O(n)$ time.
- Combine phase takes $O(n)$ time.
- Recursive call on (S_L, X_L, Y_L) takes $T(n/2)$ time; recursive call on (S_R, X_R, Y_R) takes $T(n/2)$ time.

Therefore, $T(n)$ obeys the following recurrence relation:

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 3 \\ 2T(n/2) + O(n) & \text{otherwise} \end{cases}$$

We can solve $T(n)$ using the Master Theorem to obtain $T(n) = O(n \lg n)$

Integer Multiplication

Problem

Multiply two integers x, y represented as sequences (e.g., arrays) of 0-1 bits where the lengths of the sequences can be **arbitrarily** large (assume the length of the two to be both n, with possibly padding 0's)

Notice: This cannot be simply done in constant time: the multiplication of provided by the CPU only supports a *fixed* length on the sequence (e.g., 64).

An Algorithm Everybody Knows

Solution:

- Compute a "partial product" by multiplying each digit of y separately by x, and then you add up all the partial products.
- Only this time we do the *binary* version, i.e., we multiplying each *bit* of y by x and then add up.

$$
\begin{array}{r}
12 \\
\times\,13 \\
\hline
36 \\
12 \\
\hline
156
\end{array}
\qquad\qquad
\begin{array}{r}
1100 \\
\times\,1101 \\
\hline
1100 \\
0000 \\
1100 \\
1100 \\
\hline
10011100
\end{array}
$$

(a) **(b)**

(Figure taken from [Kleinberg&Tardos - Algorithm Design])

Time complexity:

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\qquad\qquad
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$$

(a) **(b)**

(Figure taken from [Kleinberg&Tardos - Algorithm Design])

Time complexity: $O(n^2)$

Attempting to Improve the Naive Algorithm

Using, of course, divide and conquer:

Attempting to Improve the Naive Algorithm

Using, of course, divide and conquer:

- Write x as $x = x_1 \cdot 2^{n/2} + x_0$, where x_1 is the "high-order" half bits x_0 is the "low-order" half bits
- Similarly write y as $y = y_1 \cdot 2^{n/2} + y_0$

Attempting to Improve the Naive Algorithm

Using, of course, divide and conquer:

- Write x as $x = x_1 \cdot 2^{n/2} + x_0$, where x_1 is the "high-order" half bits x_0 is the "low-order" half bits
- Similarly write y as $y = y_1 \cdot 2^{n/2} + y_0$
- Rewrite xy as

$$\begin{aligned} xy &= (x_1 \cdot 2^{n/2} + x_0)(y_1 \cdot 2^{n/2} + y_0) \\ &= x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0 \end{aligned}$$

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So, to compute xy (multiplying two n-sequences), we only need to:

- Recursively compute four multiplications of $n/2$-sequences:

$$x_1 y_1,\ x_1 y_0,\ x_0 y_1,\ \text{and}\ x_0 y_0$$

- Then take the sum $x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0$ (which can be done in $O(n)$ time)

Pseudocode

Recursive-Multiply(x,y)

1. write $x = x_1 \cdot 2^{n/2} + x_0, \; y = y_1 \cdot 2^{n/2} + y_0$
2. $x_1 y_1 = \text{Recursive-Multiply}(x_1, y_1)$
3. $x_1 y_0 = \text{Recursive-Multiply}(x_1, y_0)$
4. $x_0 y_1 = \text{Recursive-Multiply}(x_0, y_1)$
5. $x_0 y_0 = \text{Recursive-Multiply}(x_0, y_0)$
6. **return** $x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0$

Time complexity:

Pseudocode

Recursive-Multiply(x,y)

1. write $x = x_1 \cdot 2^{n/2} + x_0$, $y = y_1 \cdot 2^{n/2} + y_0$
2. $x_1 y_1 =$ Recursive-Multiply(x_1, y_1)
3. $x_1 y_0 =$ Recursive-Multiply(x_1, y_0)
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6. **return** $x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0$

Time complexity:

- $T(n) = 4T(n/2) + O(n)$ which is $O(n^2)$ (no improvement at all!)

Second Attempt to Improve the Naive Algorithm

- The problem with the previous divide-and-conquer approach is that it involves **four** recursive calls
- If we can reduce the number of recursive calls to **three**, we would have

  $$T(n) = 3T(n/2) + O(n)$$

  which is $O(n^{1.59})$ (quite an improvement!)

Second Attempt to Improve the Naive Algorithm

- Notice that our goal is to compute the sum

$$xy = x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0 \tag{1}$$

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- Consider another multiplication

$$p = (x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$$

  where we observe $x_1 y_0 + x_0 y_1 = p - x_1 y_1 - x_0 y_0$

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  where we observe $x_1 y_0 + x_0 y_1 = p - x_1 y_1 - x_0 y_0$
- So, to get the three components in the sum (1), we only need the three multiplications of $n/2$-sequences:

$$x_1 y_1,\ x_0 y_0, \text{ and } p = (x_1 + x_0)(y_1 + y_0)$$

  by letting $x_1 y_0 + x_0 y_1 = p - x_1 y_1 - x_0 y_0$
- And then we can get xy with only three recursive calls!

Pseudocode (Improved)

Recursive-Multiply(x,y)

1. write $x = x_1 \cdot 2^{n/2} + x_0$ and $y = y_1 \cdot 2^{n/2} + y_0$
2. compute $x_1 + x_0$ and $y_1 + y_0$
3. $p =$ Recursive-Multiply$(x_1 + x_0, y_1 + y_0)$
4. $x_1 y_1 =$ Recursive-Multiply(x_1, y_1)
5. $x_0 y_0 =$ Recursive-Multiply(x_0, y_0)
6. **return** $x_1 y_1 \cdot 2^n + (p - x_1 y_1 - x_0 y_0) \cdot 2^{n/2} + x_0 y_0$

Time complexity:

- $T(n) = 3T(n/2) + O(n)$ which is $O(n^{1.59})$

Strassen's Algorithm for Matrix Multiplication

Problem

Given two $n \times n$ matrix $A = (a_{i,j})$ and $B = (b_{i,j})$, compute $C = A \cdot B$ which is another $n \times n$ matrix $(c_{i,j})$ with:

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

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The straightforward algorithm runs in $\Theta(n^3)$ time as we need to computer n^2 number of entries $c_{i,j}$, each takes $\Theta(n)$ multiplications and additions

A Divide-and-conquer approach

- Partition each of A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \quad B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \quad C = \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}$$

- For simplicity, assume $n = 2^k$ so that we can keep on recursively performing such partitioning into smaller matrices

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- For simplicity, assume $n = 2^k$ so that we can keep on recursively performing such partitioning into smaller matrices
- We have that $C = A \cdot B$ can be expressed as:

$$\begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \cdot \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$$

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- That is,

$$\begin{aligned} C_{1,1} &= A_{1,1} \cdot B_{1,1} + A_{1,2} \cdot B_{2,1} \\ C_{1,2} &= A_{1,1} \cdot B_{1,2} + A_{1,2} \cdot B_{2,2} \\ C_{2,1} &= A_{2,1} \cdot B_{1,1} + A_{2,2} \cdot B_{2,1} \\ C_{2,2} &= A_{2,1} \cdot B_{1,2} + A_{2,2} \cdot B_{2,2} \end{aligned}$$

Pseudocode

RecurMatMul(A,B)

let n be the number of rows on A and B
let C be a new $n \times n$ matrix
if $n == 1$:
    $c_{1,1} = a_{1,1} \cdot b_{1,1}$
    return C
partition A, B, and C each into four sub-matrices
$C_{1,1} = \text{RecurMatMul}(A_{1,1}, B_{1,1}) + \text{RecurMatMul}(A_{1,2}, B_{2,1})$
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- 4 matrix summations in line 7-10 takes $O(n^2)$ time (so other than the recursive calls it takes $O(n^2)$ time)

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- 4 matrix summations in line 7-10 takes $O(n^2)$ time (so other than the recursive calls it takes $O(n^2)$ time)
- There are 8 recursive calls each of which takes $T(n/2)$ time
- $T(n) = 8T(n/2) + O(n^2)$ which is $O(n^3)$ (no improvement at all!)

Strassen's Algorithm for Matrix Multiplication

Idea:

- Use **seven** recursive calls to multiplication of smaller matrix (instead of eight)
- Recursive equation becomes $T(n) = 7T(n/2) + O(n^2)$
- So overall complexity becomes $O(n^{\log_2 7})$ which is $O(n^{2.81})$

Step 1

Create the following 10 matrices:

$$
\begin{aligned}
S_1 &= B_{1,2} - B_{2,2} \\
S_2 &= A_{1,1} + A_{1,2} \\
S_3 &= A_{2,1} + A_{2,2} \\
S_4 &= B_{2,1} - B_{1,1} \\
S_5 &= A_{1,1} + A_{2,2} \\
S_6 &= B_{1,1} + B_{2,2} \\
S_7 &= A_{1,2} - A_{2,2} \\
S_8 &= B_{2,1} + B_{2,2} \\
S_9 &= A_{1,1} - A_{2,1} \\
S_{10} &= B_{1,1} + B_{1,2}
\end{aligned}
$$

Step 2

Recursively multiply the smaller matrices ($n/2 \times n/2$) for **seven** times:

$$
\begin{aligned}
P_1 &= A_{1,1} \cdot S_1 \\
P_2 &= S_2 \cdot B_{2,2} \\
P_3 &= S_3 \cdot B_{1,1} \\
P_4 &= A_{2,2} \cdot S_4 \\
P_5 &= S_5 \cdot S_6 \\
P_6 &= S_7 \cdot S_8 \\
P_7 &= S_9 \cdot S_{10}
\end{aligned}
$$

Step 3

Recover the smaller matrices of C using the matrices in Step 2:

$$
\begin{aligned}
C_{1,1} &= P_5 + P_4 - P_2 + P_6 \\
C_{1,2} &= P_1 + P_2 \\
C_{2,1} &= P_3 + P_4 \\
C_{2,2} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

Step 3: Further details (1)

$$
\begin{array}{l}
A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\
\qquad\qquad\qquad - A_{22} \cdot B_{11} \qquad\qquad\qquad + A_{22} \cdot B_{21} \\
\qquad\quad - A_{11} \cdot B_{22} \qquad\qquad\qquad\qquad\qquad\qquad\qquad - A_{12} \cdot B_{22} \\
\qquad\qquad\qquad\qquad\qquad - A_{22} \cdot B_{22} - A_{22} \cdot B_{21} + A_{12} \cdot B_{22} + A_{12} \cdot B_{21} \\
\hline
A_{11} \cdot B_{11} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + A_{12} \cdot B_{21} \ ,
\end{array}
$$

$$
\begin{array}{l}
A_{11} \cdot B_{12} - A_{11} \cdot B_{22} \\
\qquad\quad + A_{11} \cdot B_{22} + A_{12} \cdot B_{22} \\
\hline
A_{11} \cdot B_{12} \qquad\qquad\quad + A_{12} \cdot B_{22} \ ,
\end{array}
$$

(Figure from [CLRS])

Step 3: Further details (2)

$$
\begin{array}{lll}
A_{21} \cdot B_{11} + A_{22} \cdot B_{11} & & \\
\quad - A_{22} \cdot B_{11} + A_{22} \cdot B_{21} & & \\
\hline
A_{21} \cdot B_{11} & + A_{22} \cdot B_{21} \, , &
\end{array}
$$

$$
\begin{array}{l}
A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\
\quad - A_{11} \cdot B_{22} \qquad\qquad\qquad\qquad + A_{11} \cdot B_{12} \\
\quad\quad - A_{22} \cdot B_{11} \qquad\qquad\qquad\qquad - A_{21} \cdot B_{11} \\
- A_{11} \cdot B_{11} \qquad\qquad - A_{11} \cdot B_{12} + A_{21} \cdot B_{11} + A_{21} \cdot B_{12} \\
\hline
\qquad\qquad A_{22} \cdot B_{22} \qquad\qquad\qquad + A_{21} \cdot B_{12} \, ,
\end{array}
$$

(Figure from [CLRS])

Final comments

- Verifying the correctness of the equations in Step 3 is tedious work
- The takeaway is that Strassen has come a long way to reduce the number of smaller matrix multiplications to **seven** with a constant number of matrix additions and subtractions
  - Imaginably, finding such equations is *very hard*
- So overall we have $T(n) = 7T(n/2) + O(n^2)$ which is $O(n^{2.81})$