Case study: Understanding topology of small image patches

Tao Hou, University of Oregon

Understanding topology of small image patches

This application is documented in the following papers

- G. Carlsson, T. Ishkhanov, V. de Silva, and A. Zomorodian, On the local behavior of spaces of natural images, International Journal of Computer Vision, (76), 1, 2008, pp. 1-12.
- Carlsson, Gunnar. "Topology and data." Bulletin of the American Mathematical Society 46.2 (2009): 255-308.

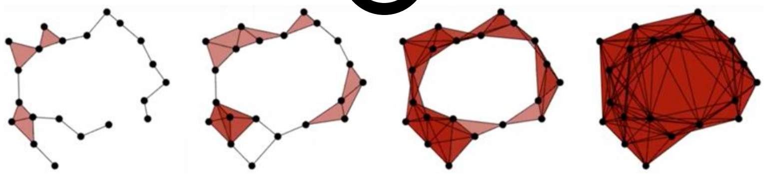
Persistent homology pipeline

1

Point cloud
(image patches)

Build Filtration

2



Čech/Rips Filtration

Persistent Homology

3

barcode

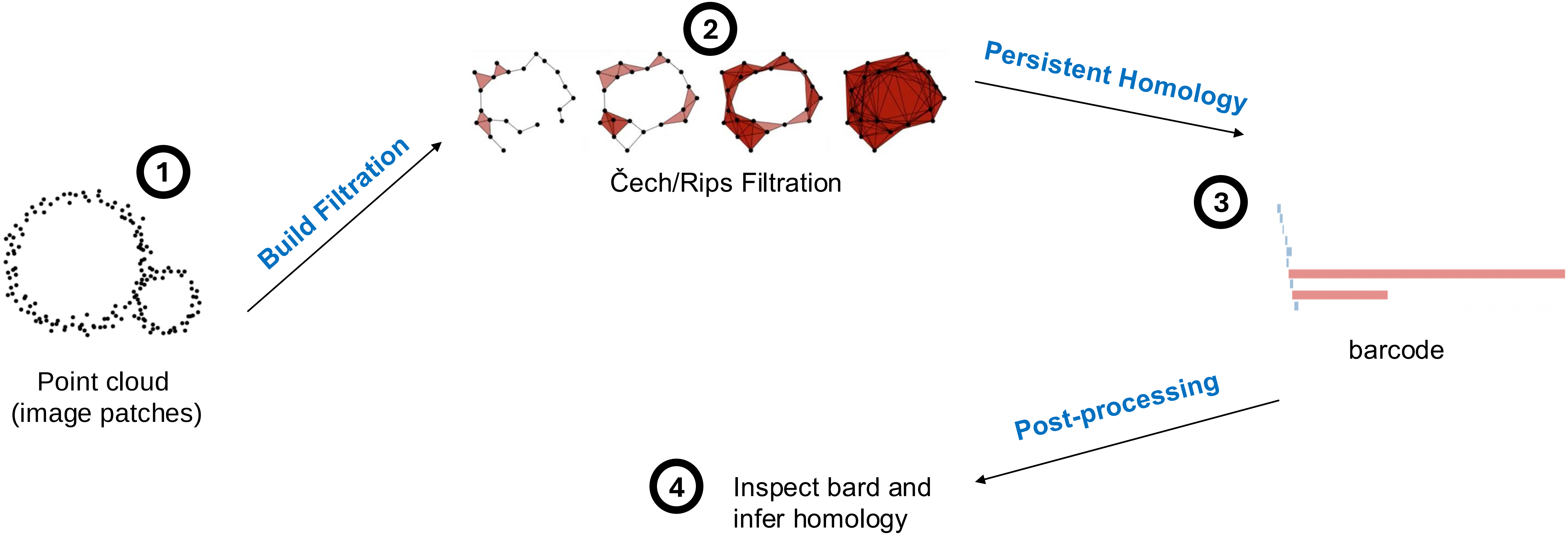
Post-processing

4

Inspect bard and
infer homology

Some img from: AATRN

Persistent homology pipeline



This paper indeed adopts an “iterative” process, aka they repeat the above processes several times to refine their inference

Some img from: AATRN

Background

- An image (assuming gray-scale) taken with a digital camera consists of a number (0-255) attached to each pixel
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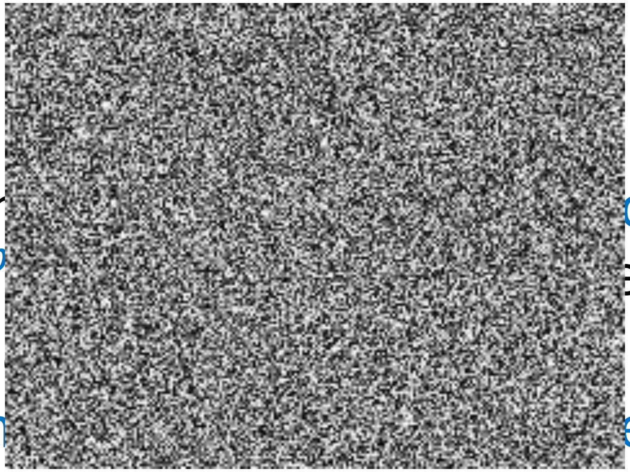
- It can be considered as a  the number of pixels (e.g., $p = 16 \times 16 = 256$)
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- The database consisted of images taken around Groningen, Holland, in town and in the surrounding countryside

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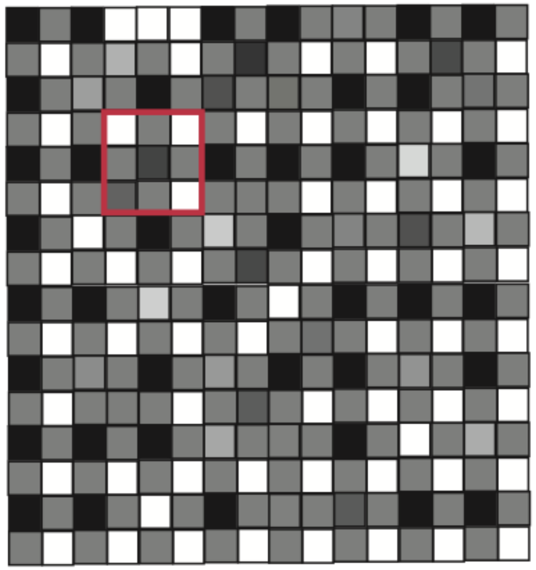
- From images from the previous database, they collect 4 millions of 3×3 patches



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   People often compute logarithms of values when processing data because it allows them to transform skewed data into a more normal distribution, making it easier to analyze using statistical methods

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Doing the above equates two patches ***which only differ by the brightness*** *aka. if a patch is obtained from another patch by adding a constant value, i.e. "turning up the brightness", then the two patches will be regarded as the same*

Data preprocessing

Images of the same scene with different brightness

Image from: https://theailearner.com/2019/01/30/what-is-contrast-in-image-processing/

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A "norm" is a generalization of the ***length*** *of a vector. Divide the vector (patch) by its "length" (D-norm) make it of length 1, so the contrast of all images is always at the same level*

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For reference, a "1-dimensional sphere" S^1 within the 2-dimensional space is the just the boundary of unit ball (consisting of all points with length 1). Reasoning the above in detail is beyond the scope

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  - The previous “normalization process” based on the D-norms make the patches to reside in a “7-dimensional sphere” S^7 within the 8-dimensional space
- They then observe that all patches are scattered throughout the 7-sphere, in the sense that no point on the 7-sphere is very far from the each other

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- But this may not be the most efficient way: a more efficient way is to consider **density** of points:
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  - Points from sparse areas should have a higher precedence because of its scarcity

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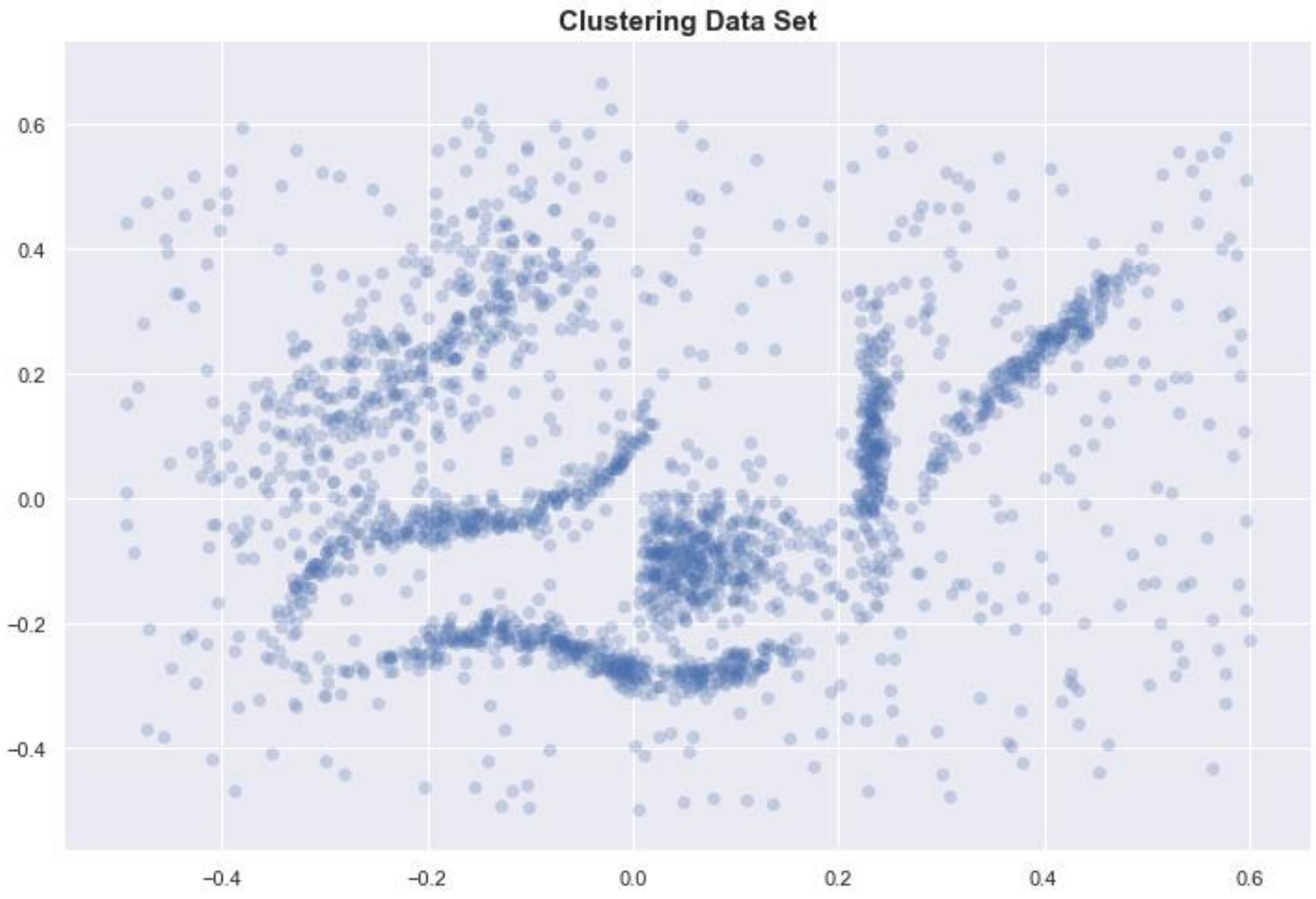

Image from: https://pberba.github.io/stats/2020/07/08/intro-hdbscan

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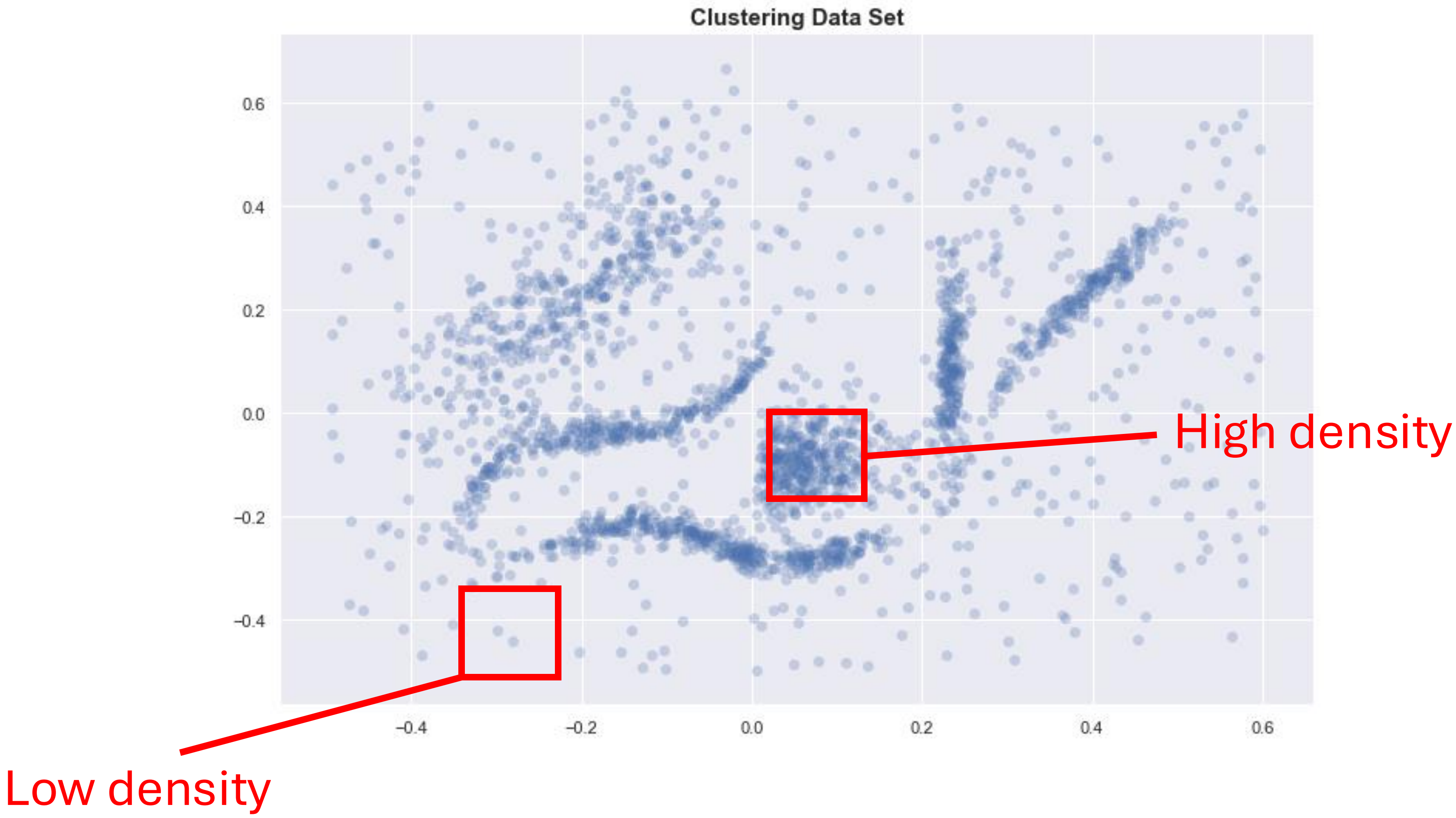

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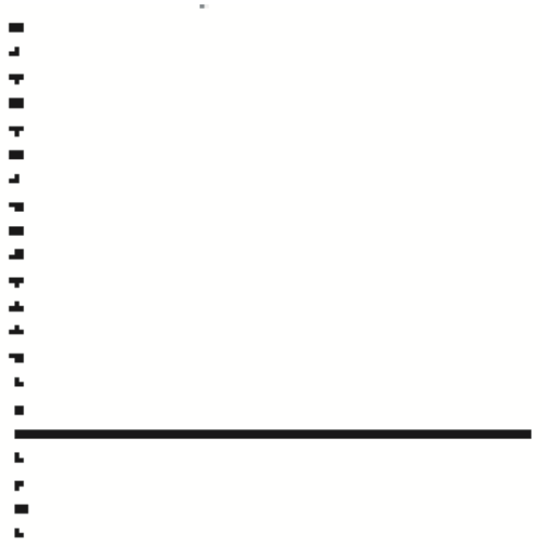
- They randomly select a certain number of points (called "landmarks") from a certain $M[k, T]$
- From the landmark points, they then build the Čech/Rips Filtration and compute the persistence barcode for the it

First attempt

- 1d barcode for 50 landmarks from $M[300,30]$:



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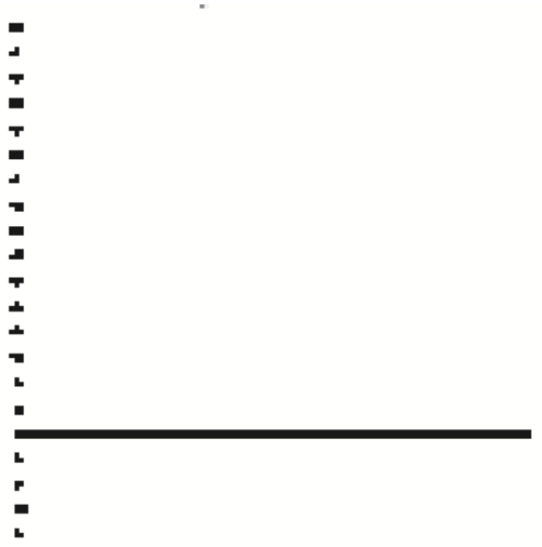

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- There are a number of short lines, and one long one
- This suggests that the first Betti number should be estimated to be one, aka. there is a single 1-d cycle (hole) across all patches (points)

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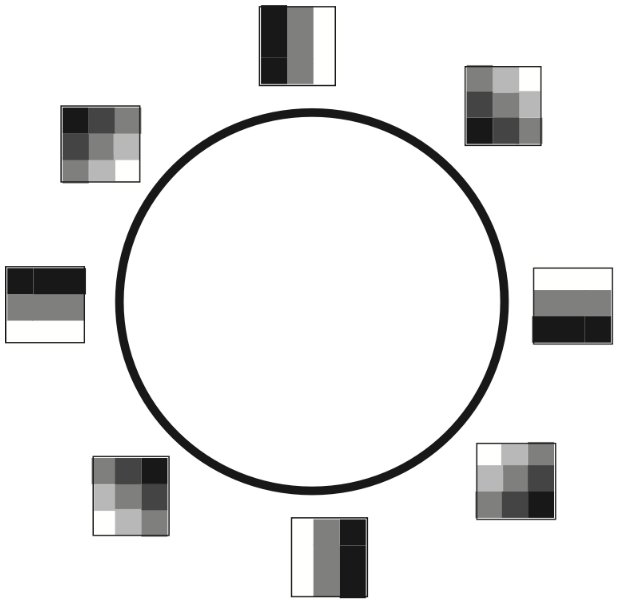

Primary circle

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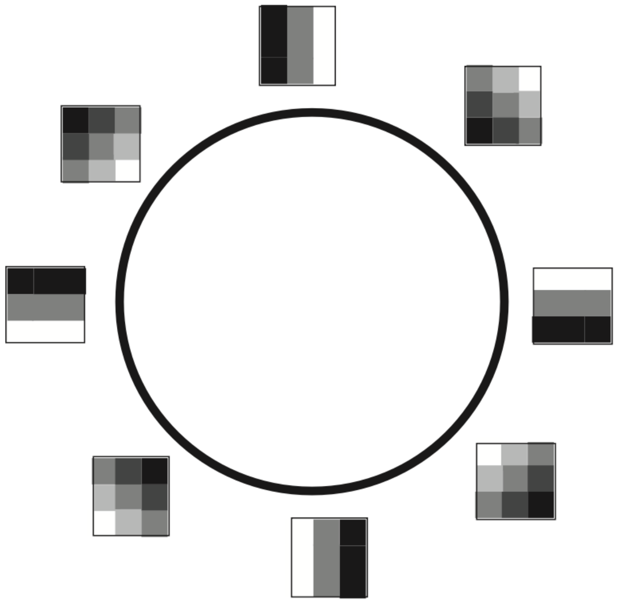
- An explanation for this is that the patches are "variation across a single direction" where the direction rotates which form a cycle



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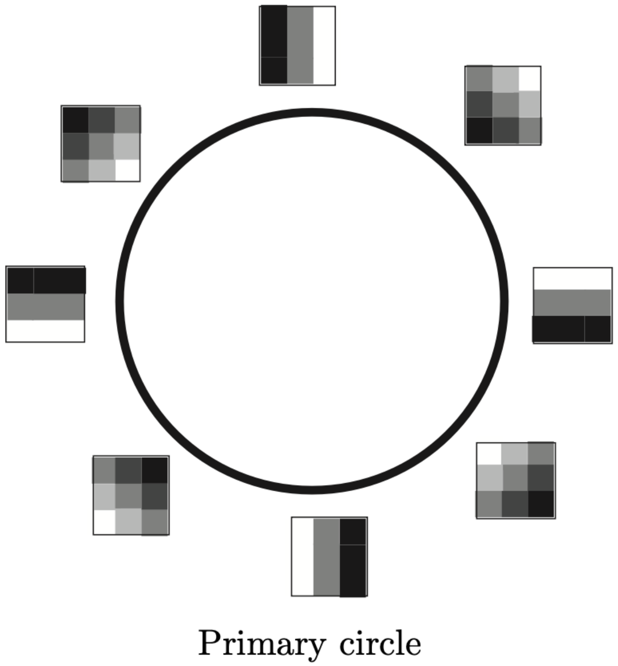

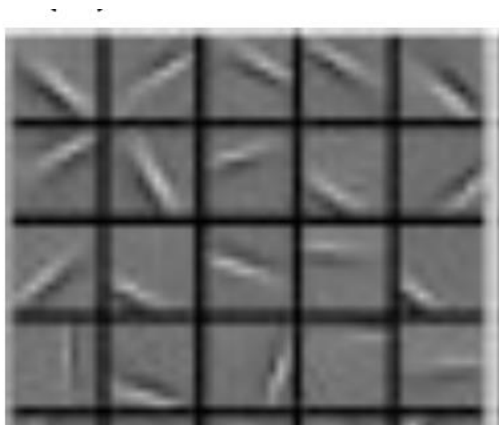

Right image from: van Hateren and van der Schaaf. Independent component filters of natural images compared with simple cells in primary visual cortex

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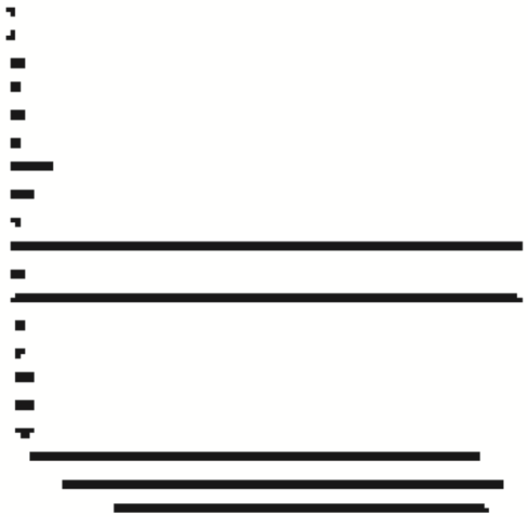

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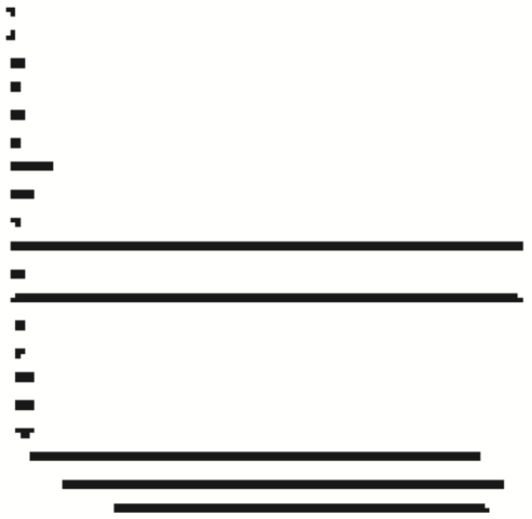

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- Notice: $\delta_k(p)$ for large k computes density using points in large neighborhoods of p, and for small k uses small neighborhoods
- So, δ_k for large k corresponds to a smoothed out notion of density, and for small k carries more of the detailed structure of the data set

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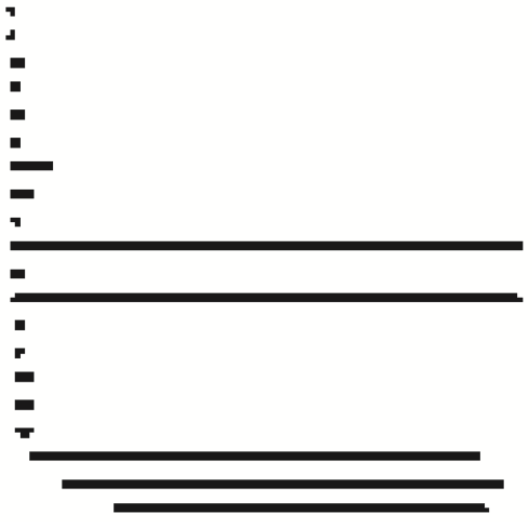
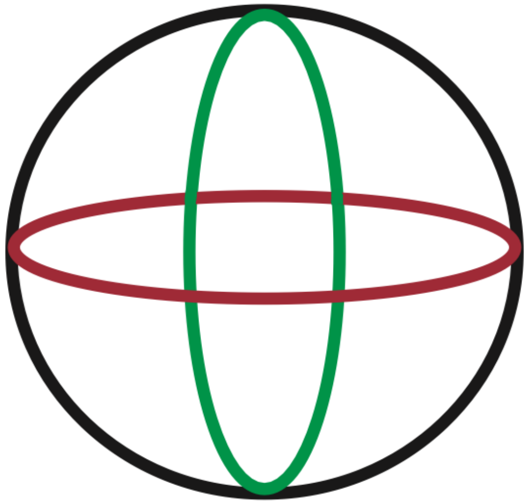


$k = 15, T = 30\%$

- There are a number of short lines, and five long ones, which is stable across different samples
- This suggests that the first Betti number should be estimated to be five

Second attempt

- The most probably space for a first Betti number of five is a space of three cycles (we will not really touch on why)



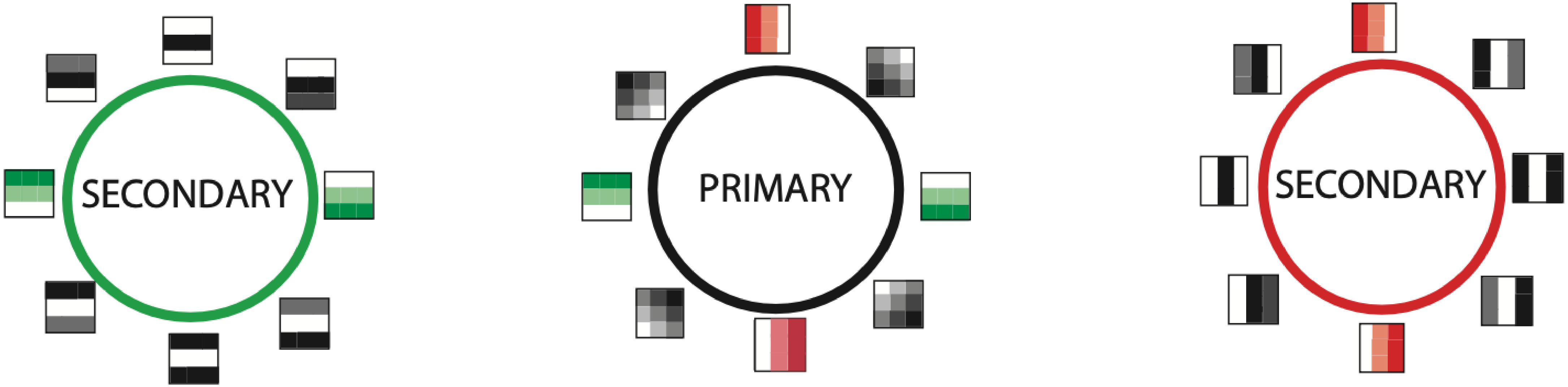
Three circle model

- Notice that the primary black cycle touches both the two secondary cycles (red and green), but the two red and green cycles do not actually touch each other

Second attempt

An explanation:

- The primary cycle still corresponds to “variation across a single direction” where the direction rotates
- The two secondary cycles capture vertical or horizonal variations of patches where the variation direction changes horizontally or vertically



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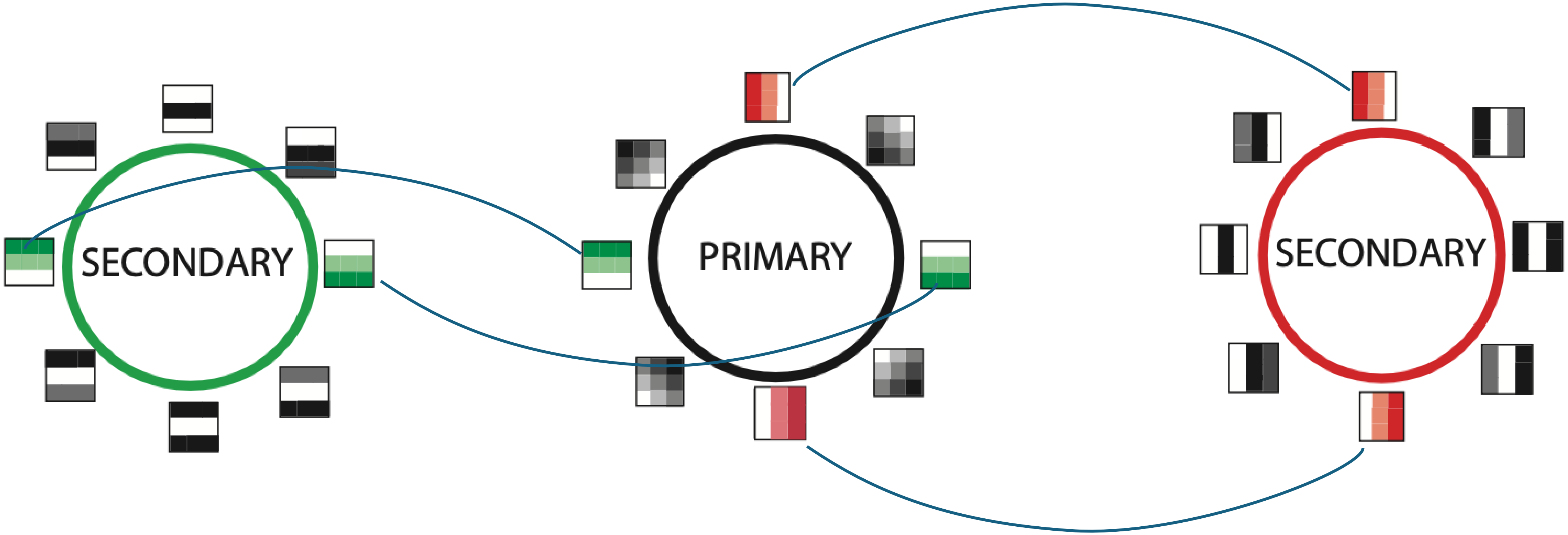
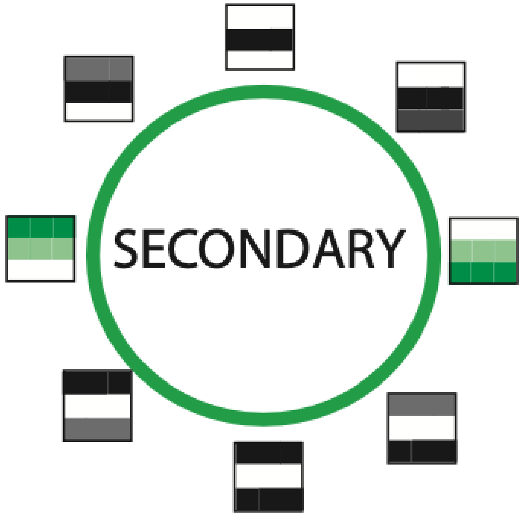
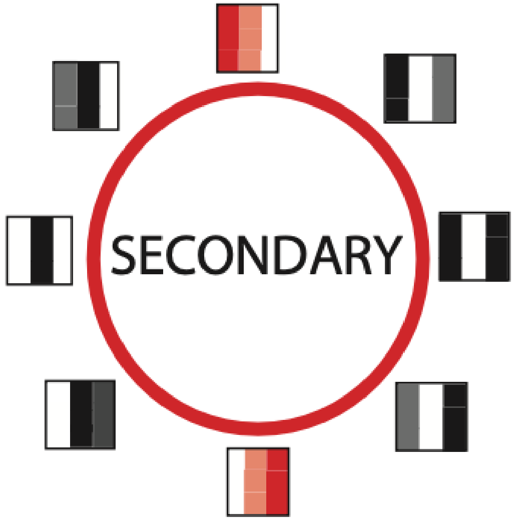


Second attempt

Explanation for vertical and horizontal variations:

1. Nature has this bias, since for example objects aligned in a vertical direction are more stable than those aligned at a 45 degree angle

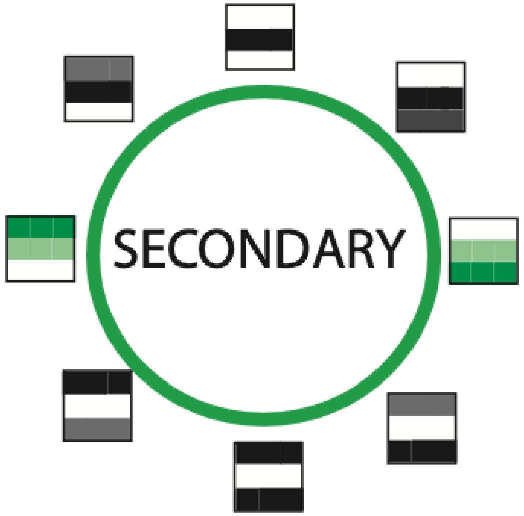


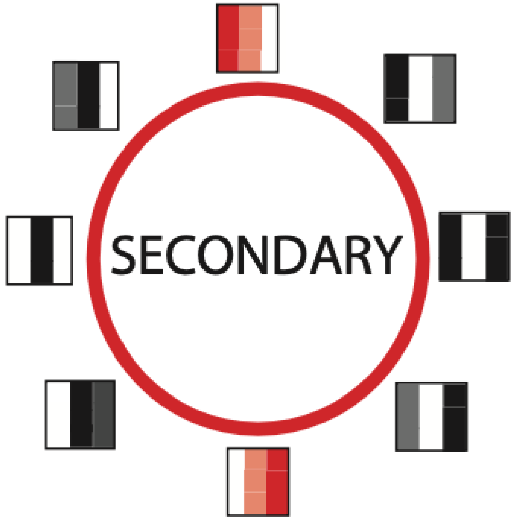


Second attempt

Explanation for vertical and horizontal variations:

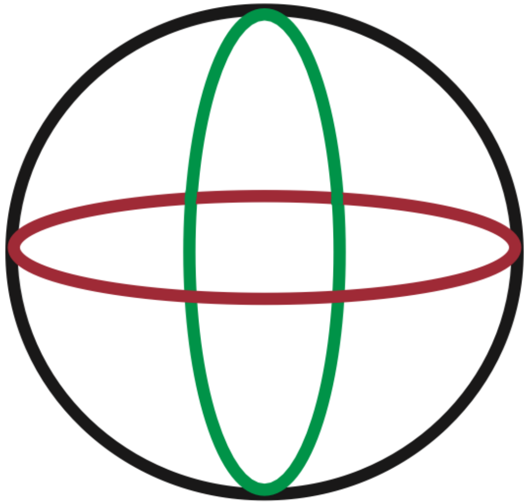
1. Nature has this bias, since for example objects aligned in a vertical direction are more stable than those aligned at a 45 degree angle
2. Another explanation is that this phenomenon is related to the technology of the camera, since the rectangular pixel arrays in the camera have the potential to bias the patches in favor of the vertical and horizontal directions





A further quest

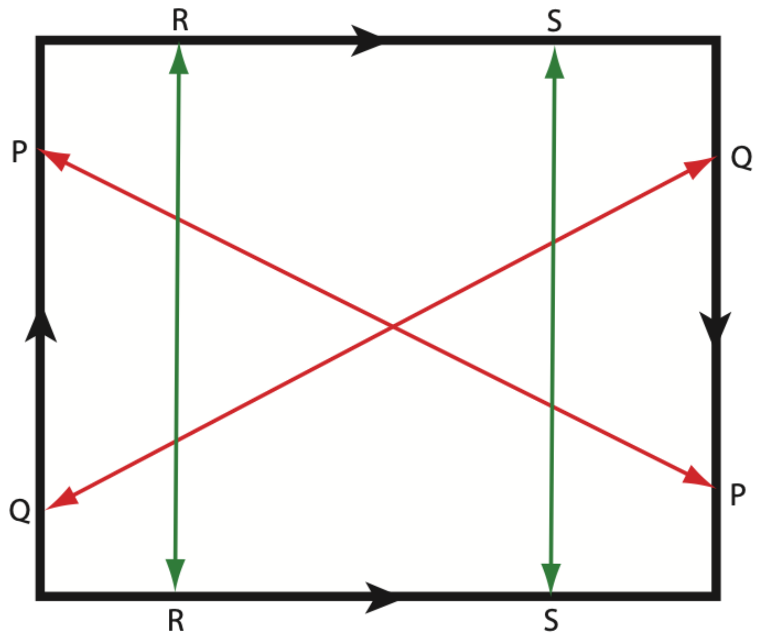
- A question to ask as an extension is that where is the three-cycle-with-two-intersections space lie in?
- Aka. if you fill out the “missing pieces” of the points, what kind of space woud you get?



Three circle model

A further quest

- A natural guess is the famous space in topology world called the "Klein bottle"
- It's a space derived by identifying sides of the following square with
  - Horizontal sides identified normally
  - Vertical sides identified **reversely**



A further quest

- Klein bottle can only be visualized in 4D world but we are in 3D
- The following is a 3D presentation which is not perfect (has self intersections)

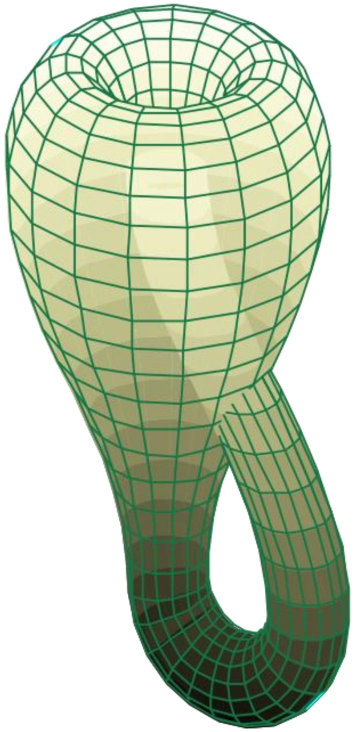
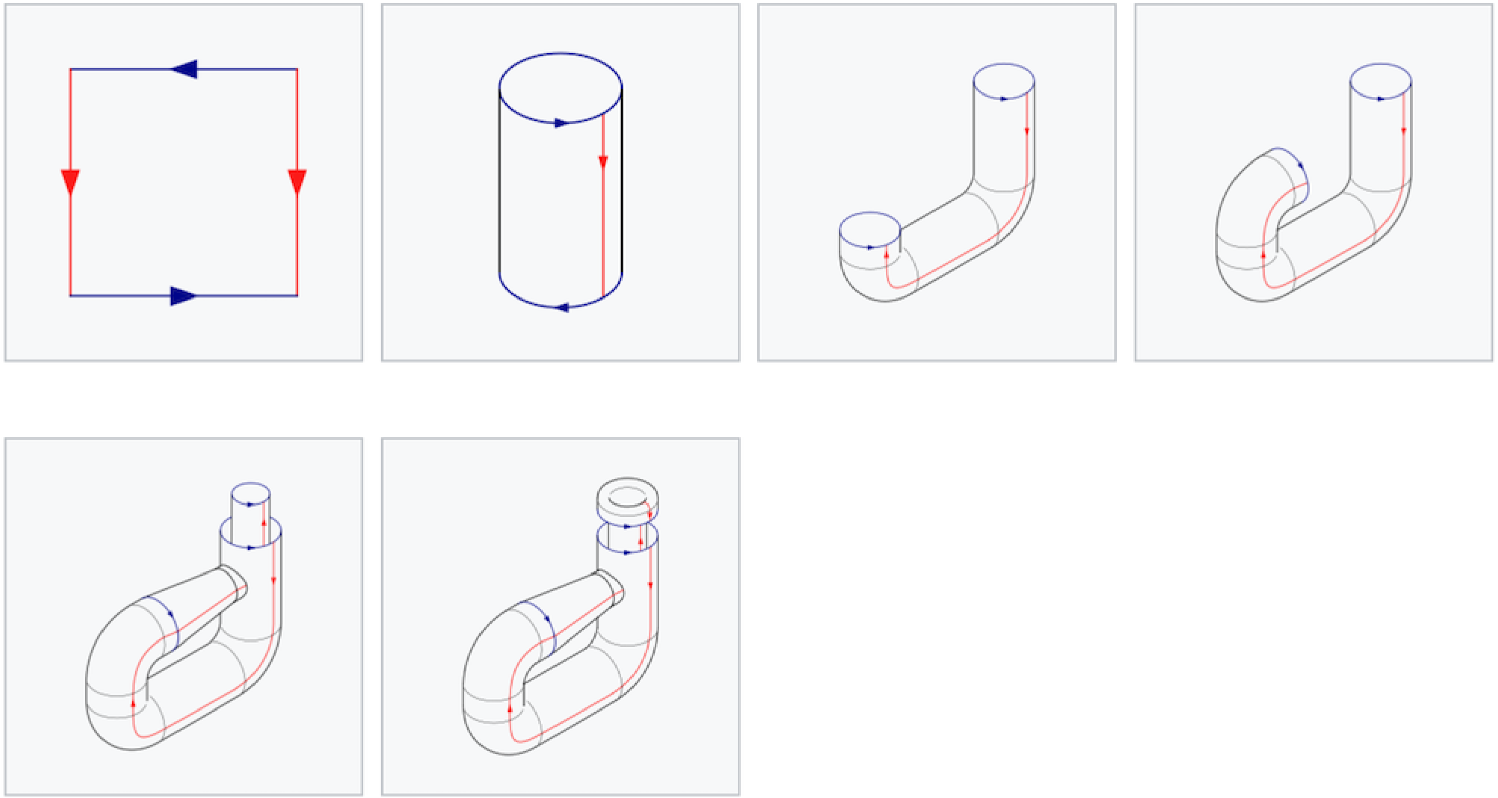


Image from: Wikipedia

A further quest

- The following is an illustration for how to identify the sides of the square to from the bottle (see also: https://plus.maths.org/content/introducing-klein-bottle)



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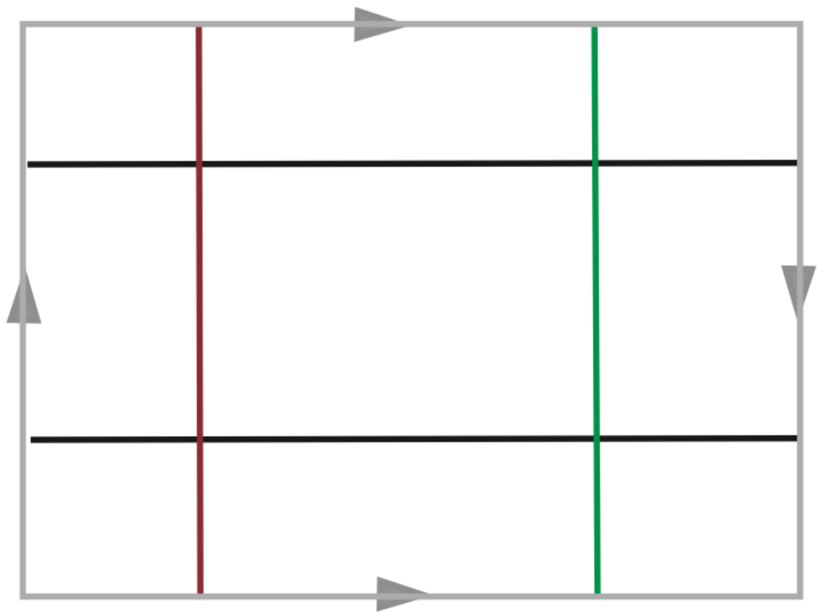

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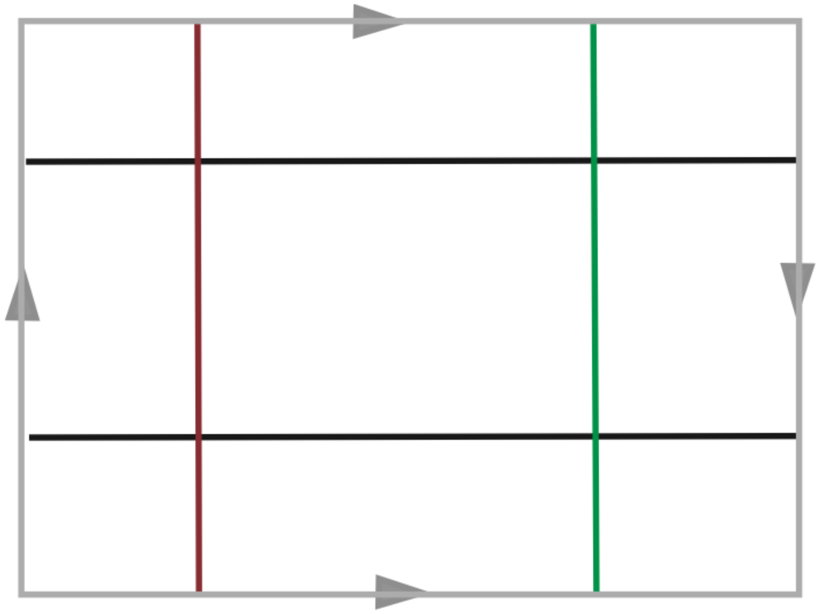

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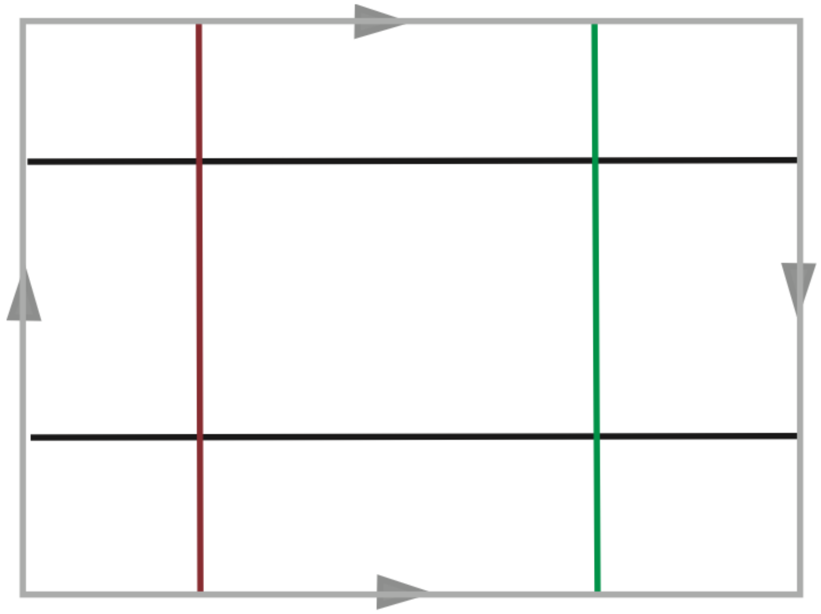
- By the identification of the sides, we notice that the the following black lines form a single cycle (the primary one)
- The two red and green lines form two separate cycles
- We also have that black line intersects red and green lines twice while red and green do not intersect (which suits our assumption)



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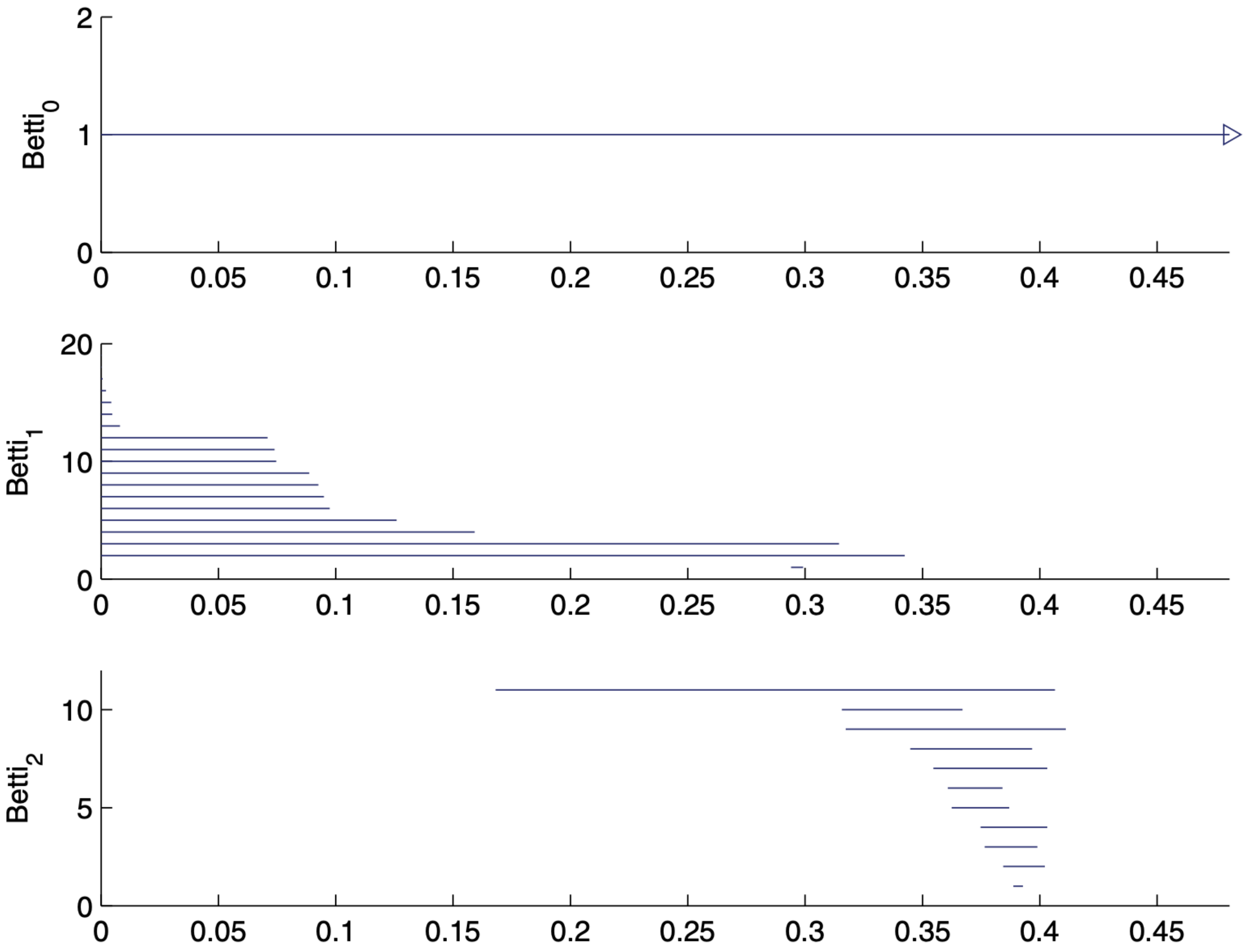
- This means that they must construct landmarks whose barcode reflect these Betti numbers
- However, after several attempts, they could not do this
- The authors suspects there are certain types of patches of missing which make them fail to “construct” the Klein bottle
- After some analysis of the nature of the patches, they find the missing patches.
- After adding them, the barcode of the landmarks finally exhibit the desired property

A further quest



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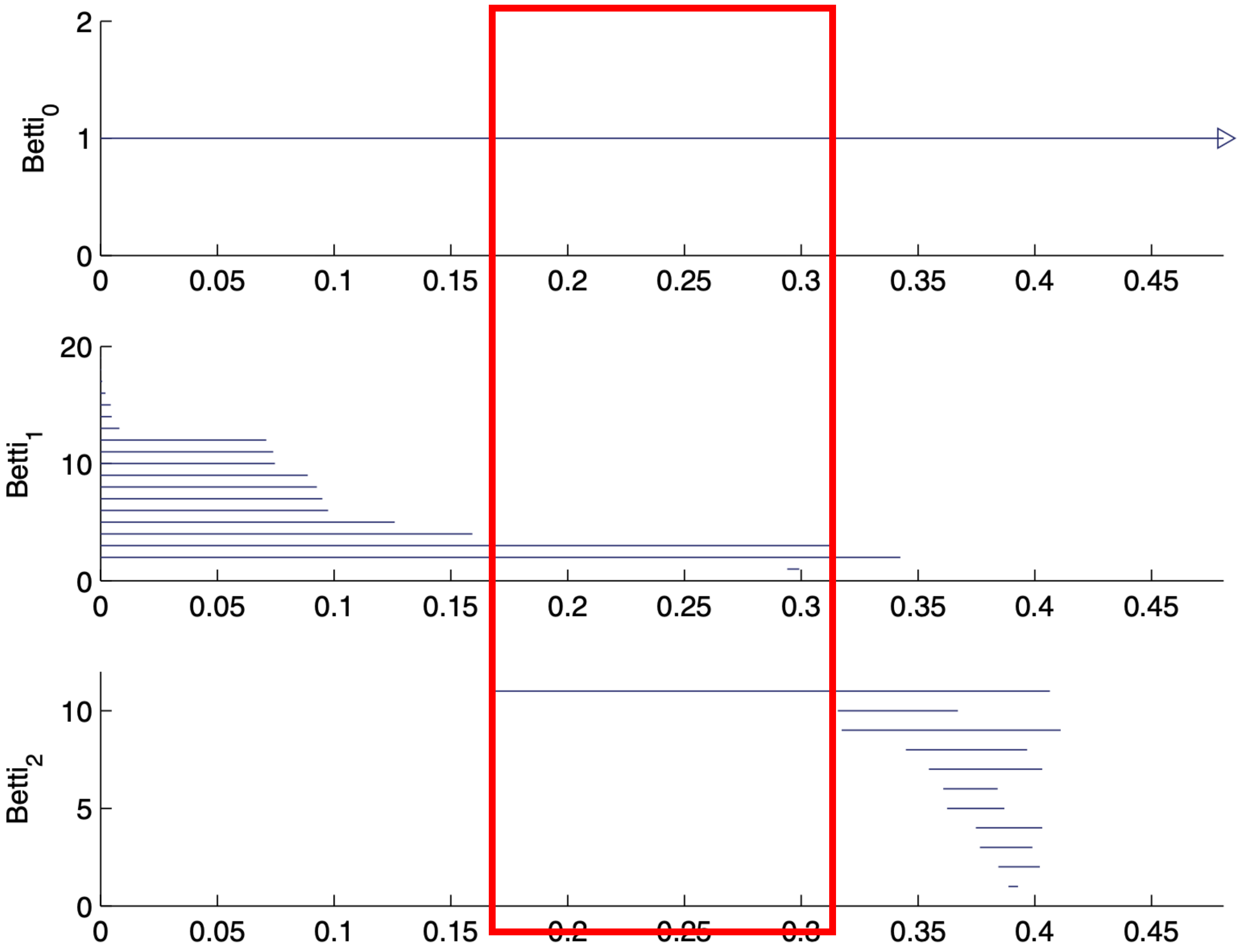

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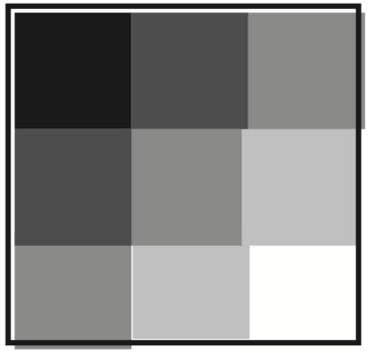
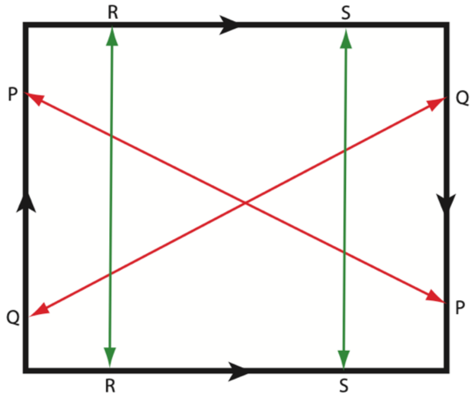


$\mathbb{R}^9$



$\mathbb{R}^2$

Further Remark

- The demonstrated case study is a almost “textbook” direct application of persistent homology
- The take a dataset (point cloud in this case), do some preprocessing and sampling, and then build the filtration and compute PD/barcode
- The whole process contains a lot of back-and-forth, trial-and-error
- They also try to refine their hypothesis or further test their findings based on initial findings