Persistent Homology: Interpretation and Stability

Tao Hou, University of Oregon

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- We also looked at several common ways for building filtration out of data that are practical in different applications
- Notice that our ultimate goal is to use persistent homology to infer the “shape” of data (i.e., homology inference)
- To do this, we need to fully understand the meanings of PD or barcode, from different aspects
- Moreover, we shall also see some properties that are critical to showing that persistent homology is a “reliable” way for inferring the shape

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- Why above the diagonal?

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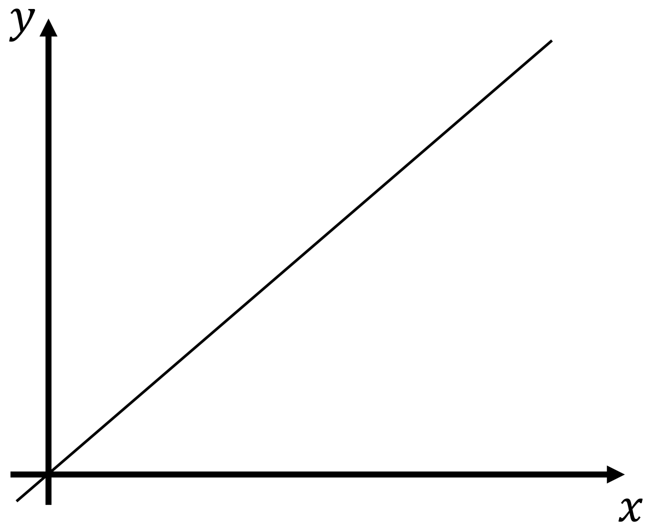

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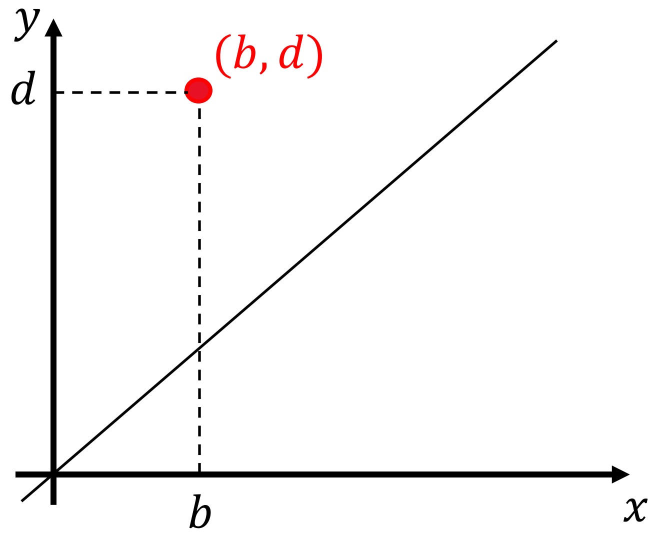

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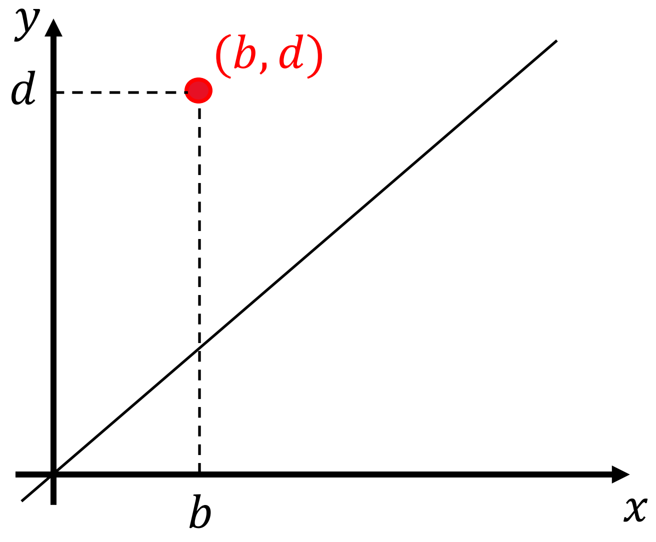

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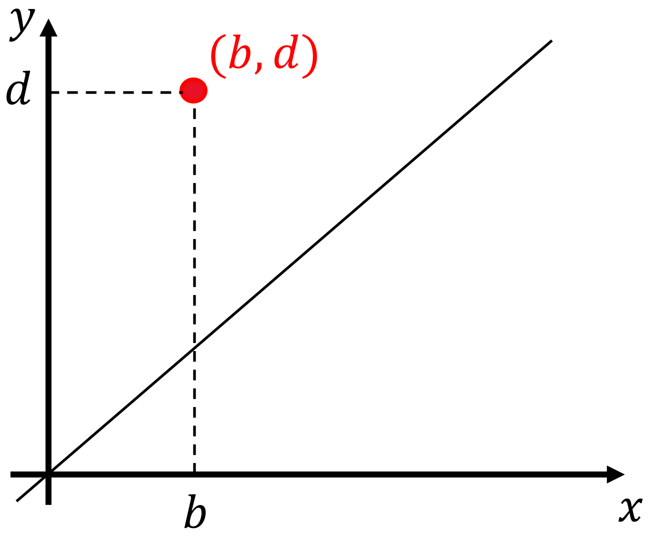

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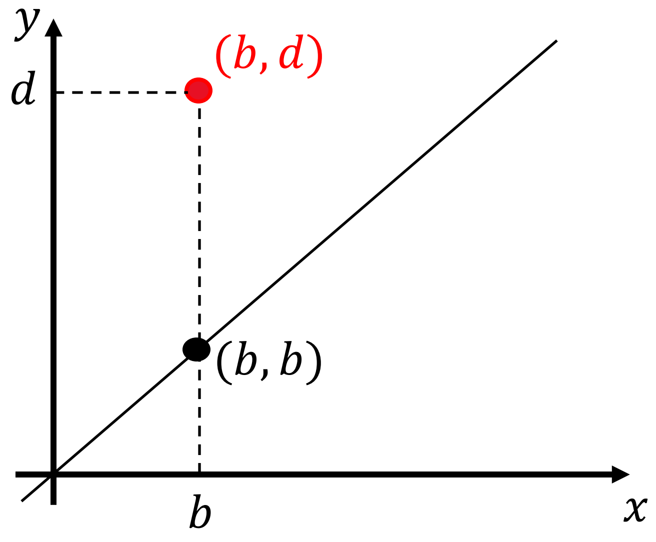

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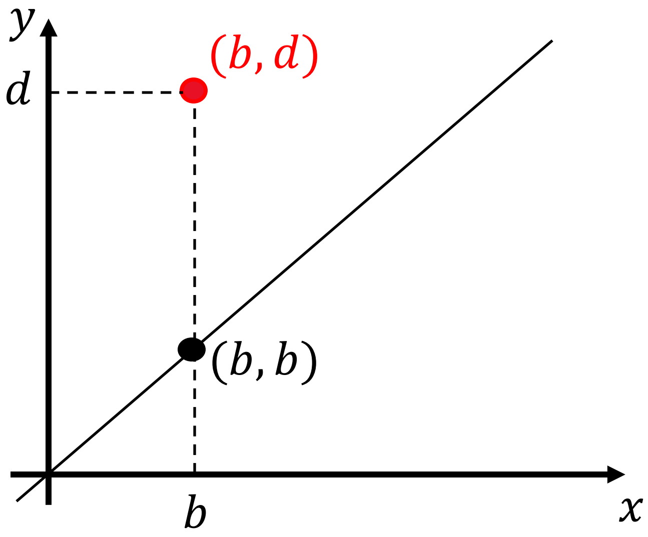

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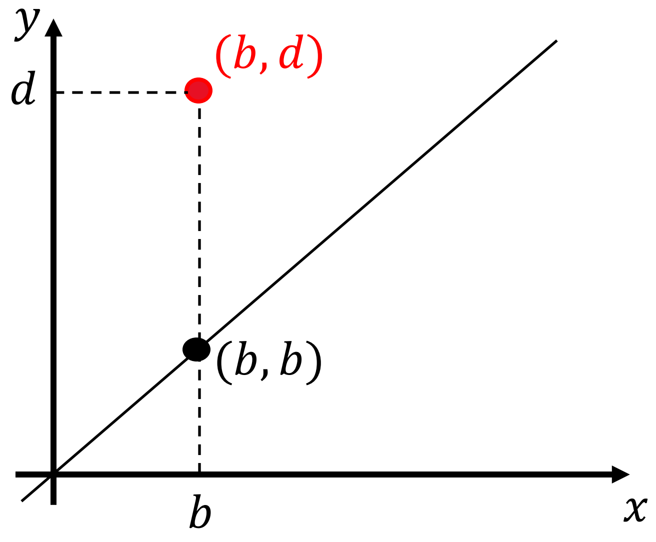
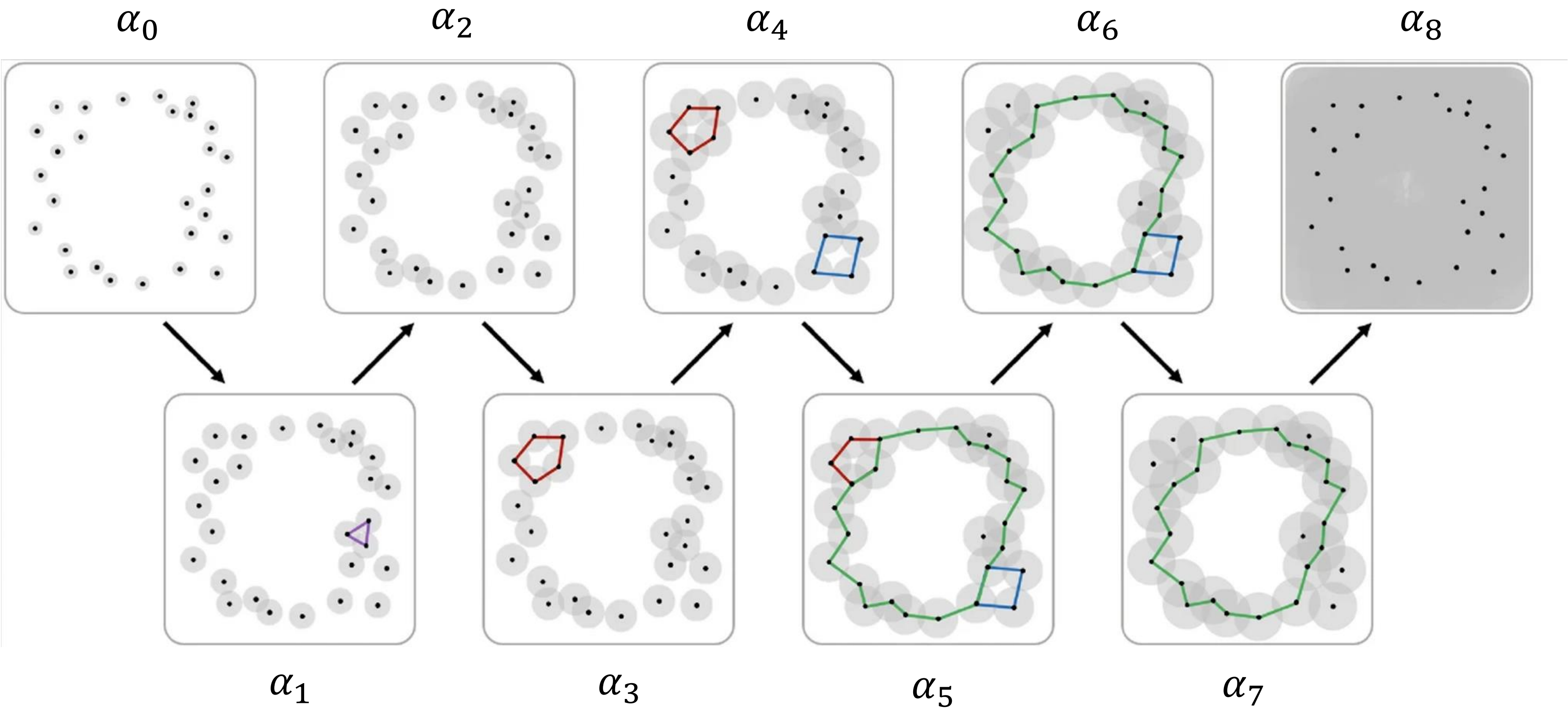


Meaning of "**length**" of a PD interval



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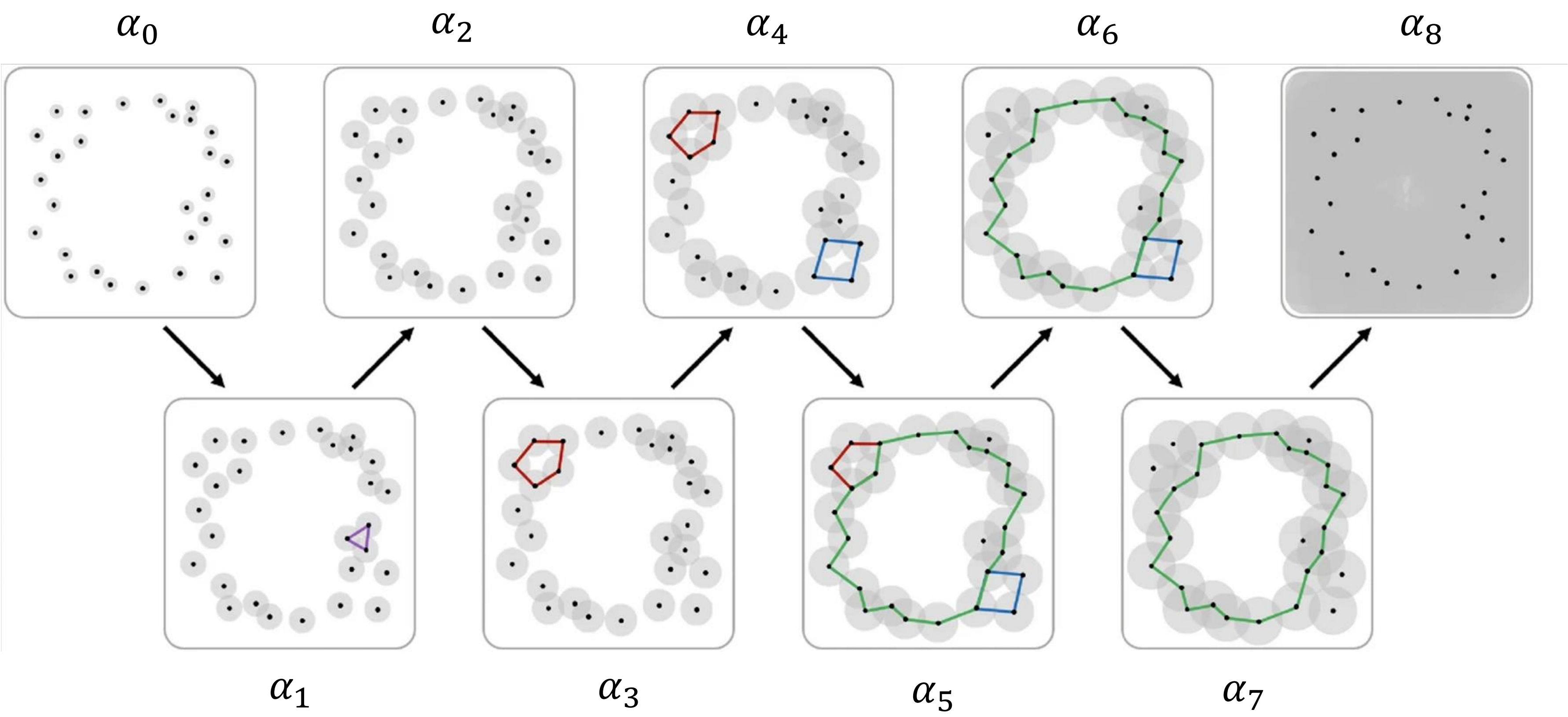

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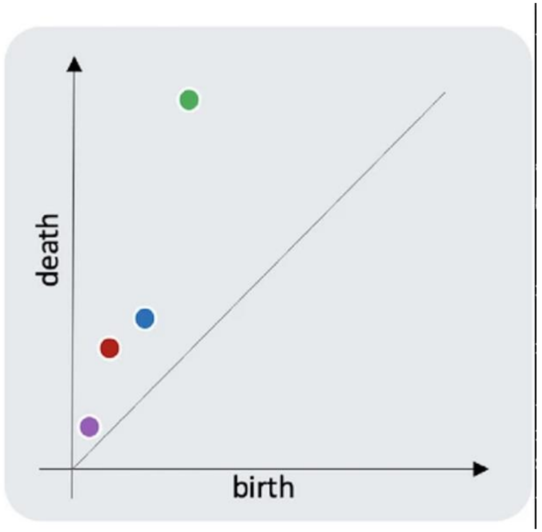

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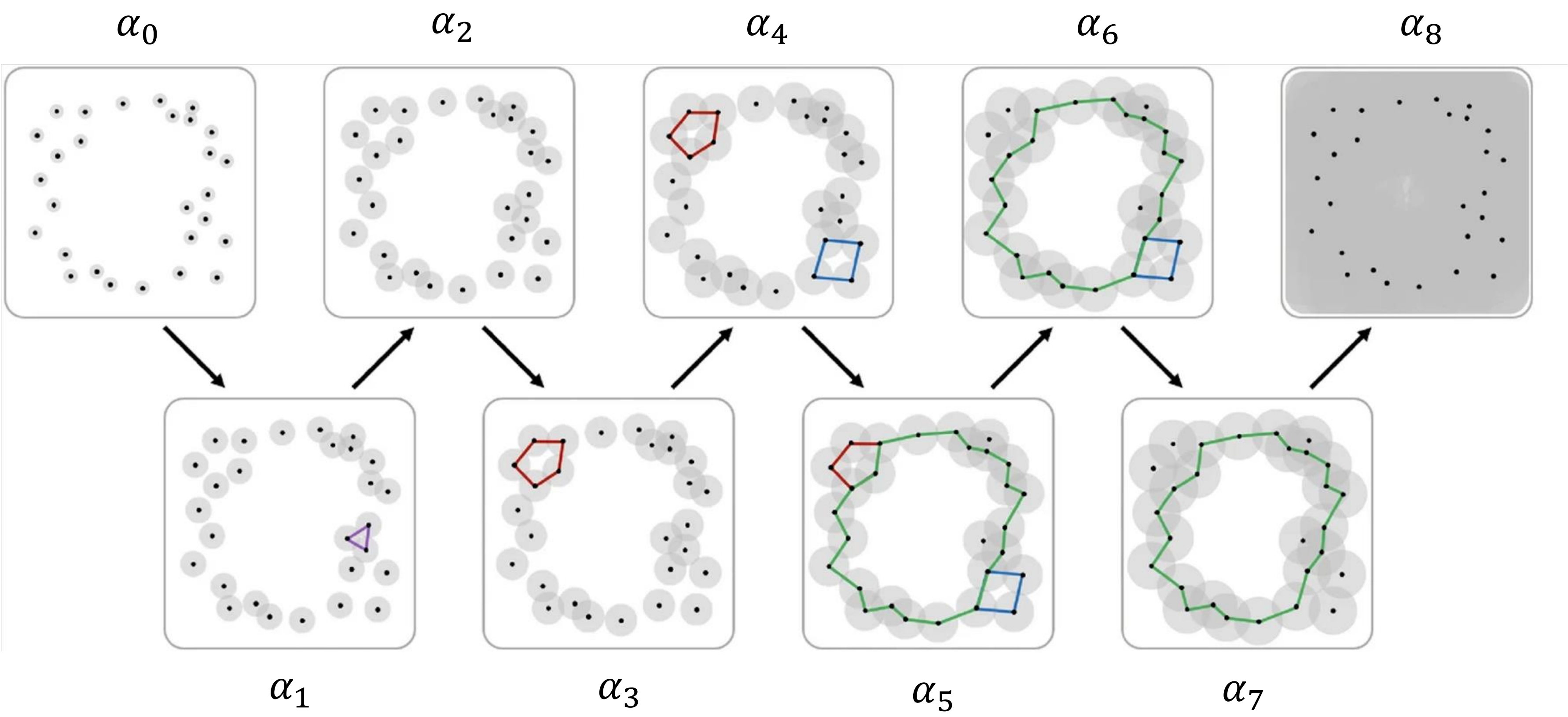

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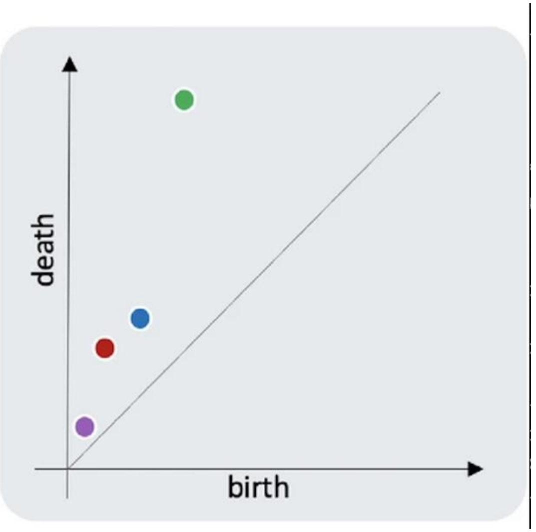

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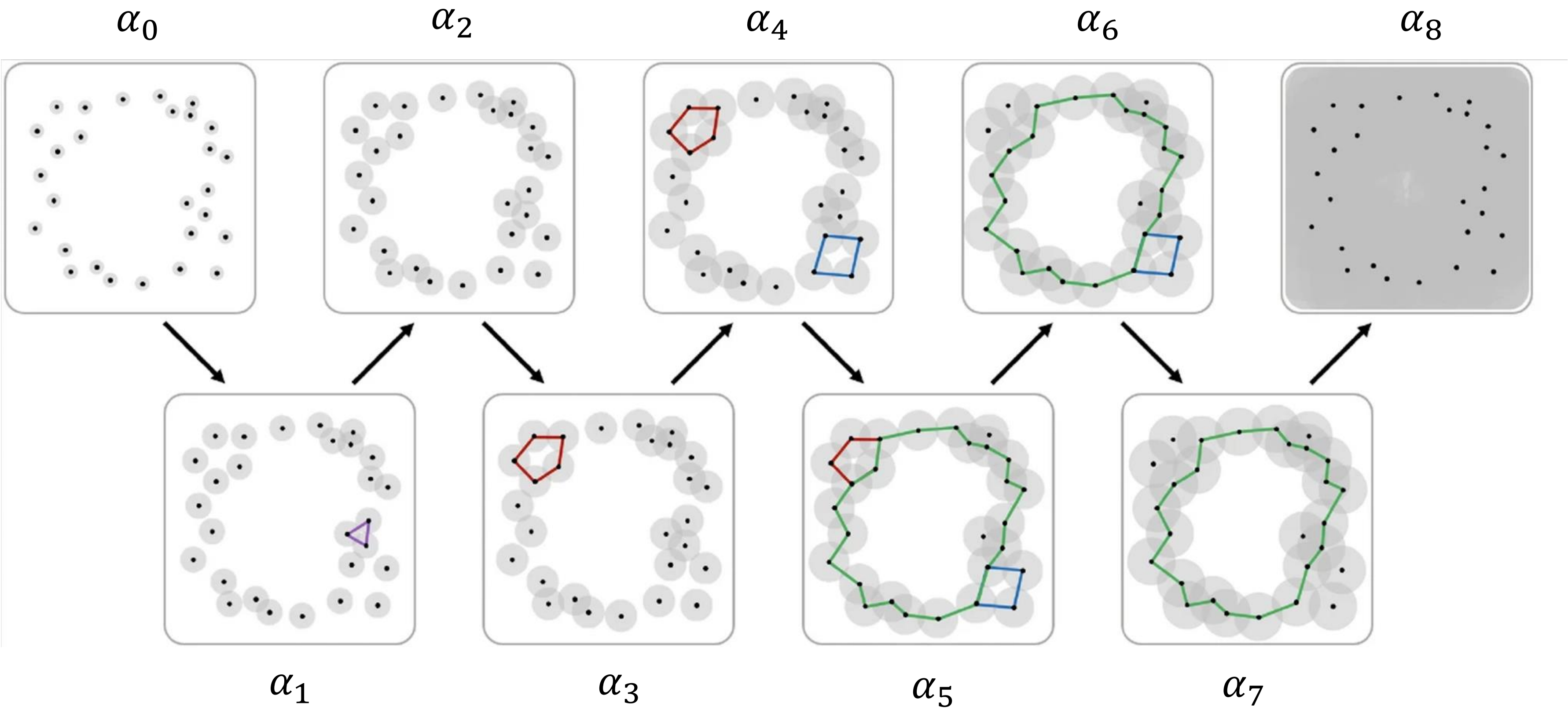
- Define length of a PD interval (point), (b, d), as $d - b$
- Observe the longer an interval is, the more significant its homology hole is



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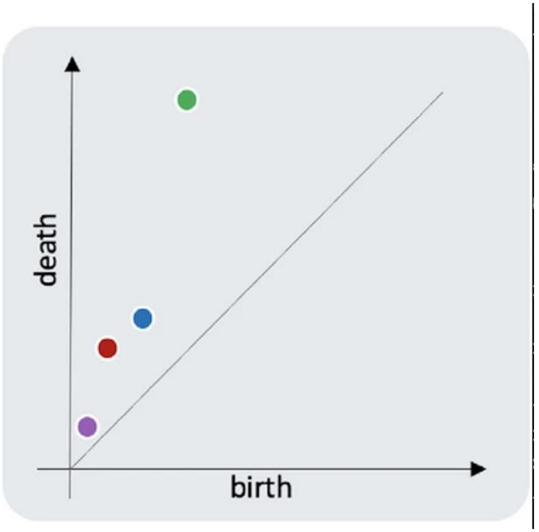

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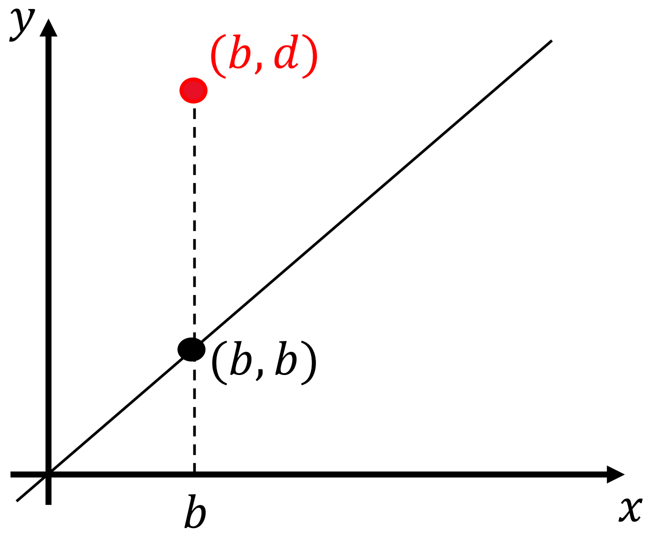

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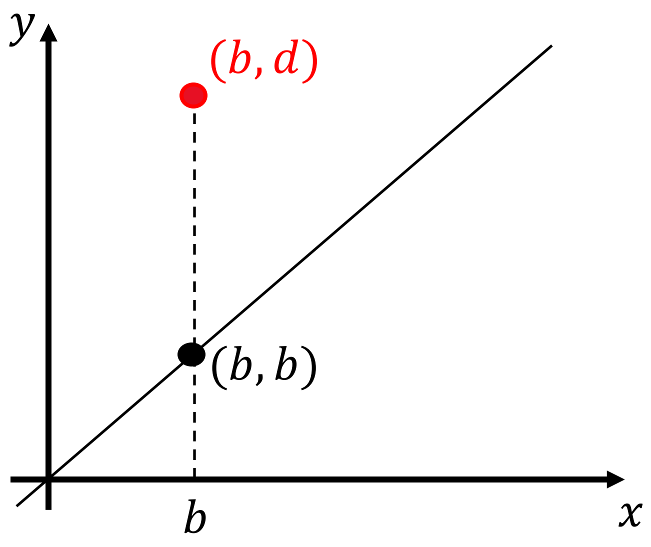
- This means that **the distance of a point in PD to the diagonal indicates the length of the corresponding interval**, and hence **the significance of the homological feature**



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- For this, we need a way to measure:
  - Difference between the data
  - Difference between two PDs
- Mathematically, the tool to measure the difference of two objects is called a **distance** function (or **metric**)

Distance function (as on Wikipedia)

Definition [edit]

Formally, a **metric space** is an ordered pair (M, d) where M is a set and d is a **metric** on M, i.e., a function

$$d : M \times M \to \mathbb{R}$$

satisfying the following axioms for all points $x, y, z \in M$:[4][5]

1. The distance from a point to itself is zero:

   $$d(x, x) = 0$$

2. (Positivity) The distance between two distinct points is always positive:

   $$\text{If } x \neq y \text{, then } d(x, y) > 0$$

3. (Symmetry) The distance from x to y is always the same as the distance from y to x:

   $$d(x, y) = d(y, x)$$

4. The triangle inequality holds:

   $$d(x, z) \leq d(x, y) + d(y, z)$$

   This is a natural property of both physical and metaphorical notions of distance: you can arrive at z from x by taking a detour through y, but this will not make your journey any shorter than the direct path.

If the metric d is unambiguous, one often refers by abuse of notation to "the metric space M".

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  - L_∞-distance: $d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

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- What is the data we try to measure here?
- It turns out that "functions" are a quite universal type of data (reason will be made clear later)
- Specifically, for measuring the difference of two functions, we assume the domain to be the same, aka. we measure two functions of the following form:
  - $f: X \to \mathbb{R}$
  - $g: X \to \mathbb{R}$

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- The distance is also denoted $\parallel f - g \parallel_\infty$:

$$\parallel f - g \parallel_\infty = \max_{x \in X}\{|f(x) - g(x)|\}$$

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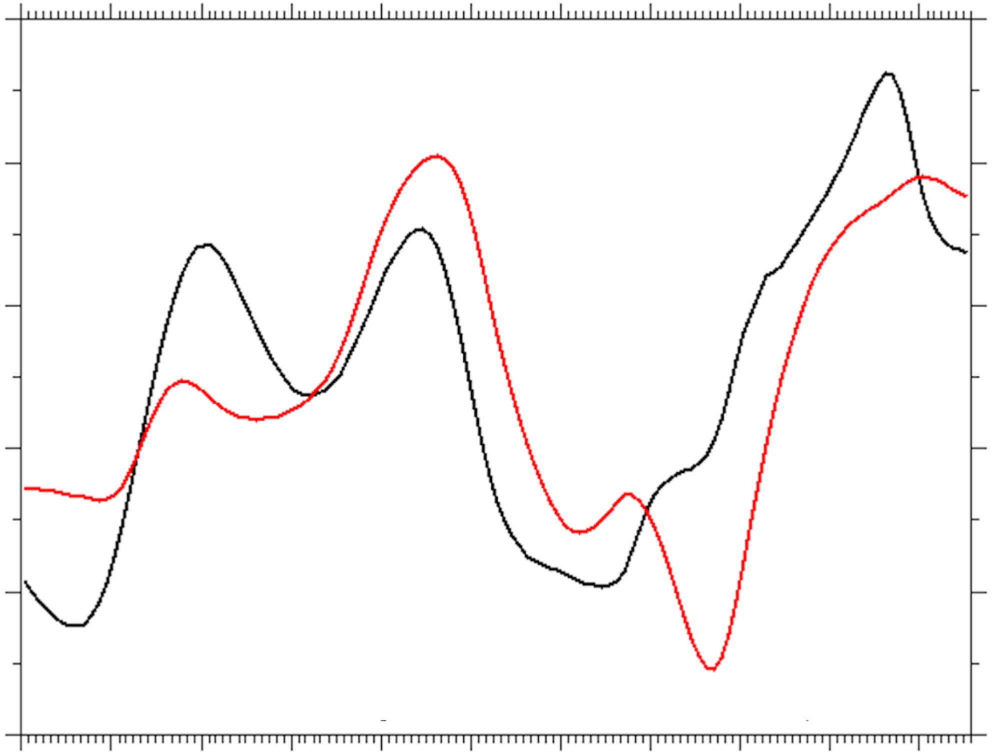
- Suppose $X = \mathbb{R}$ for f and g.



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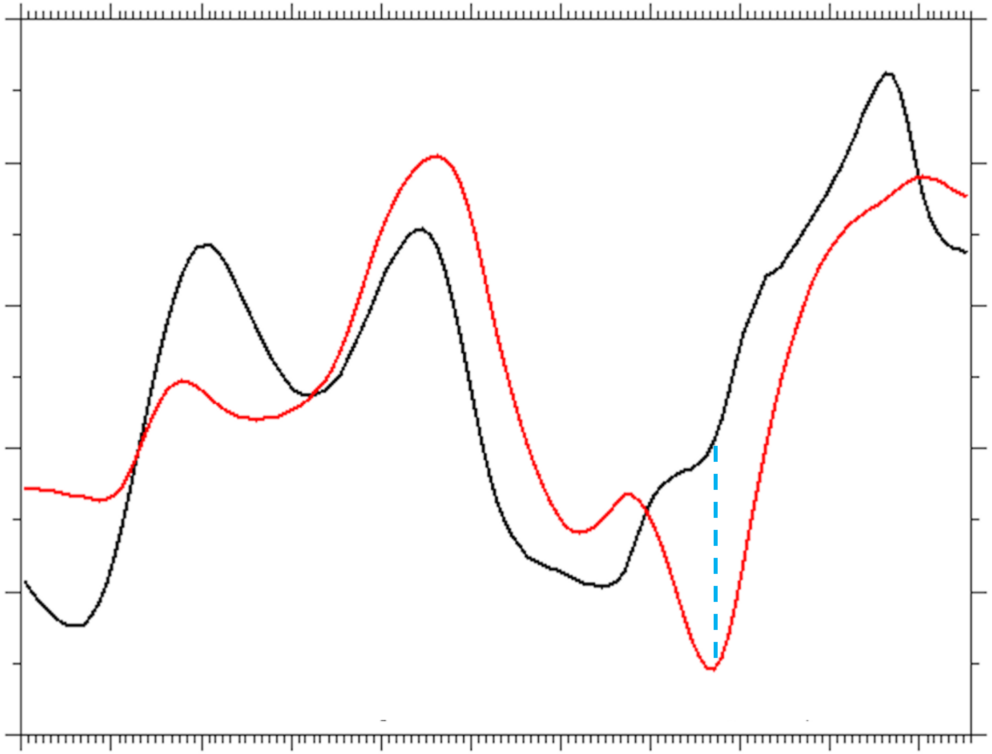

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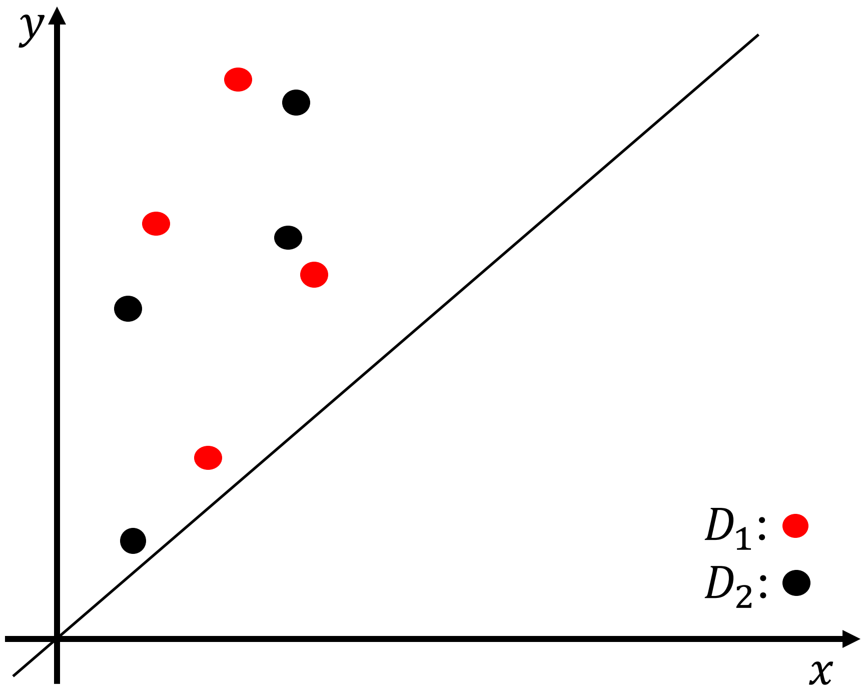

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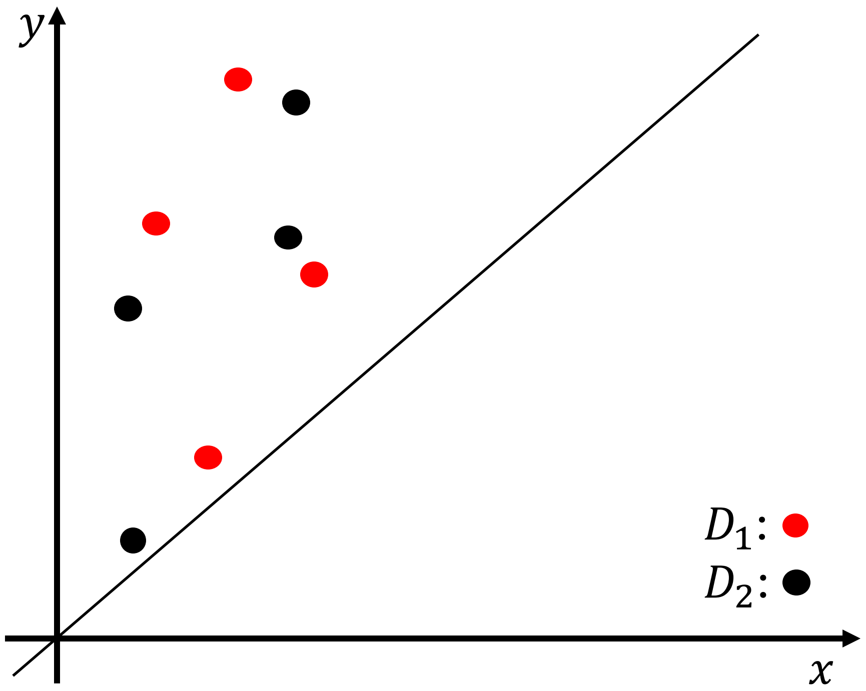

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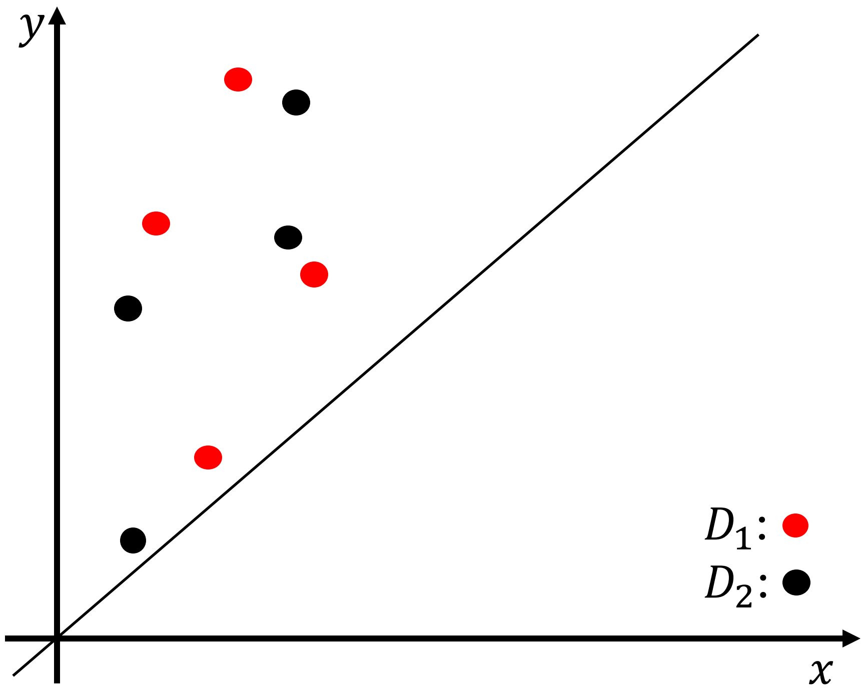

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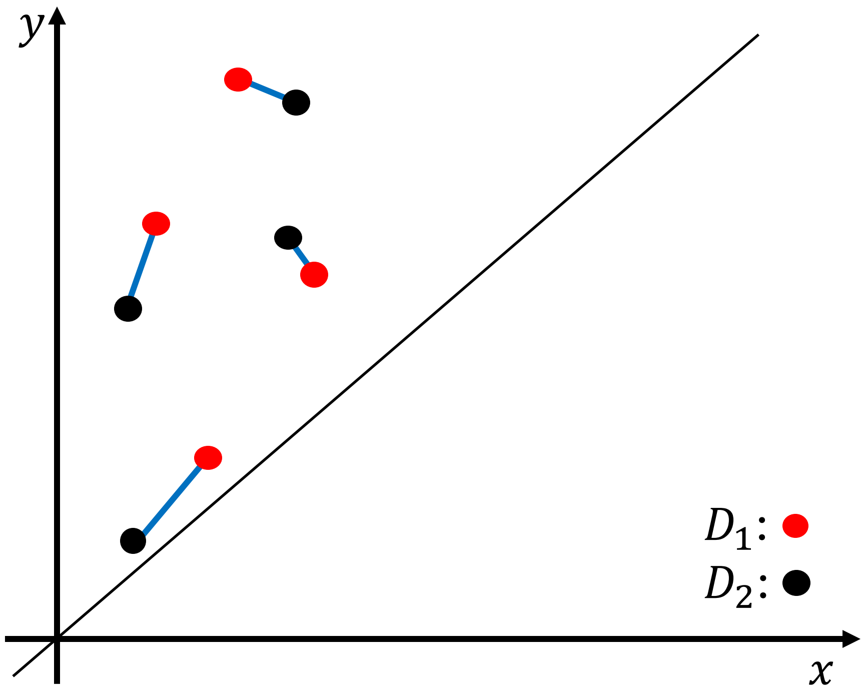

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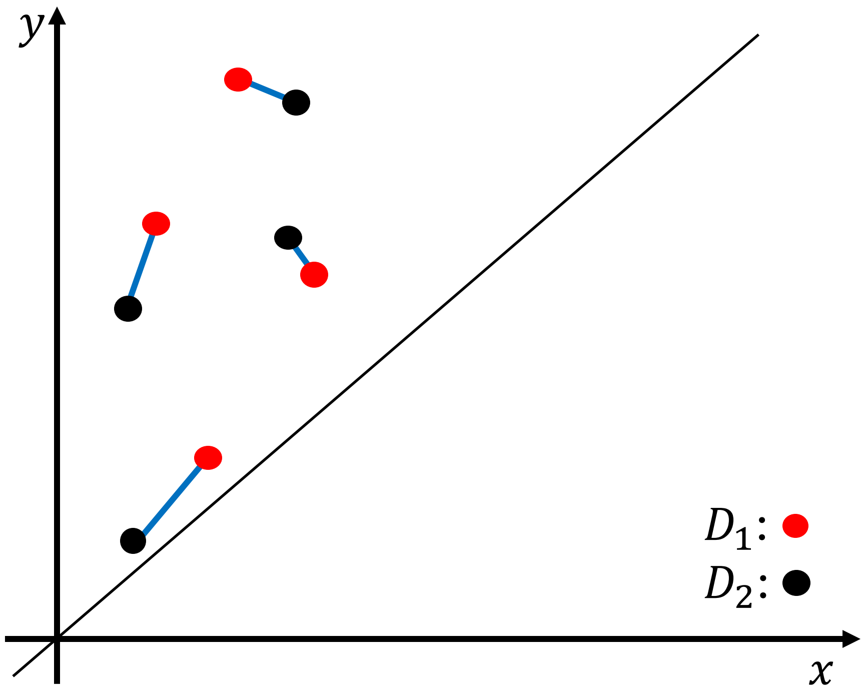

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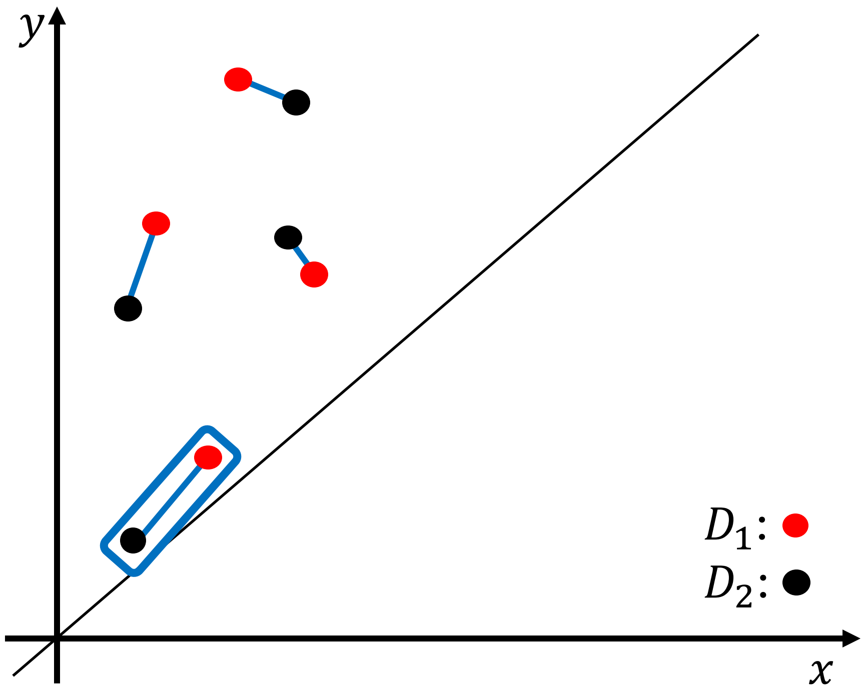

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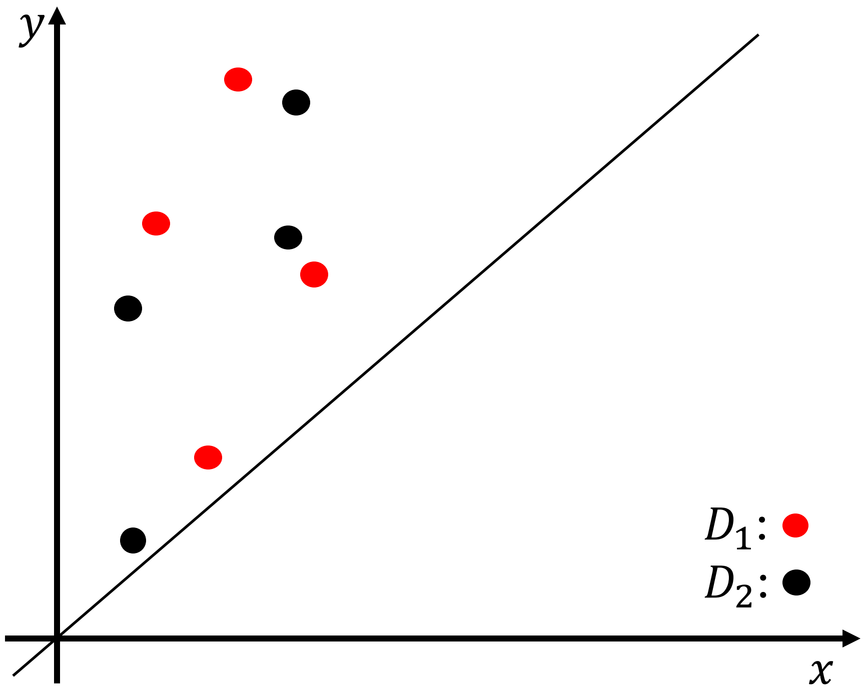

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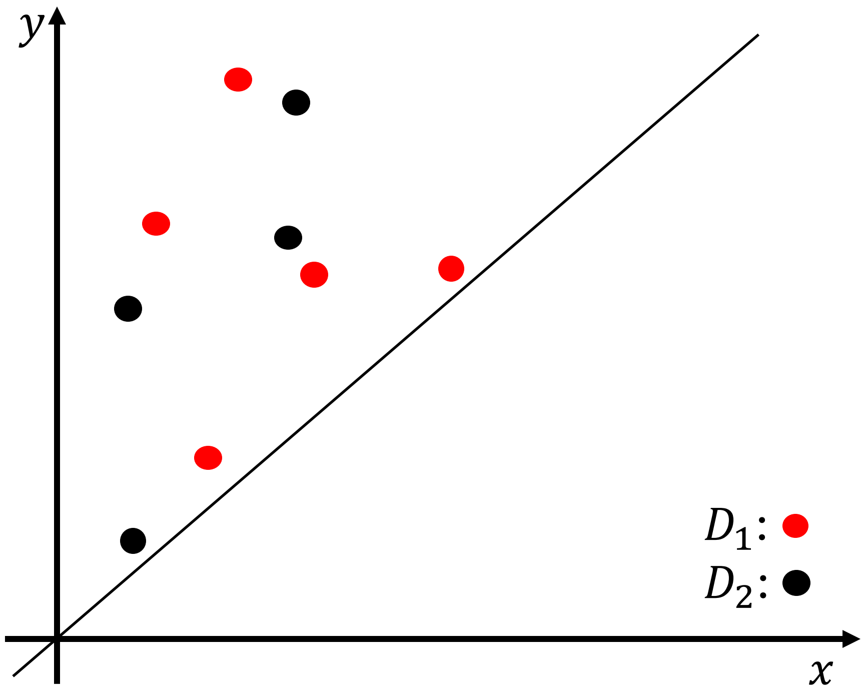

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     - However, we can only "ignore" then when they are not very "important", aka. being close to the diagonal

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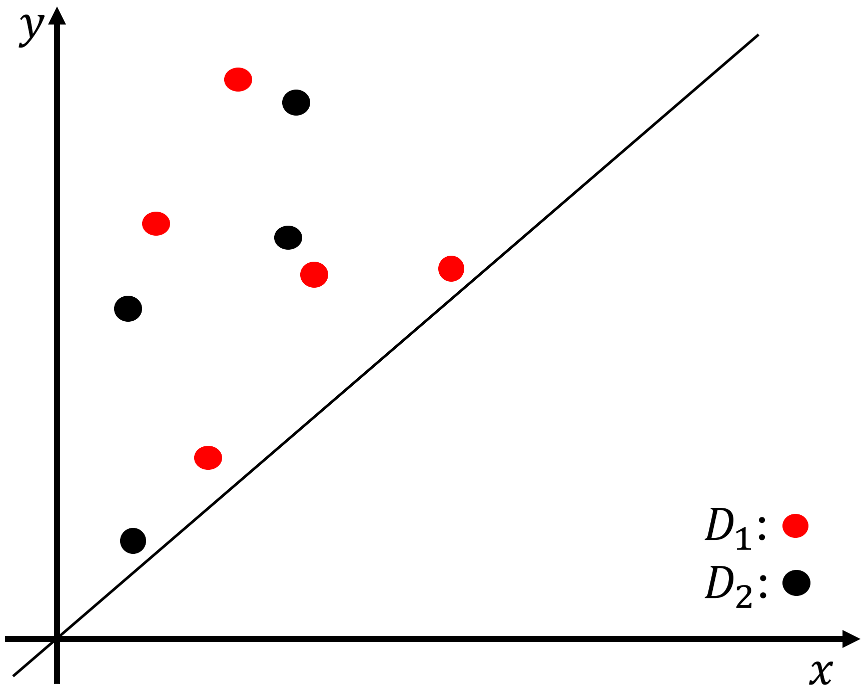

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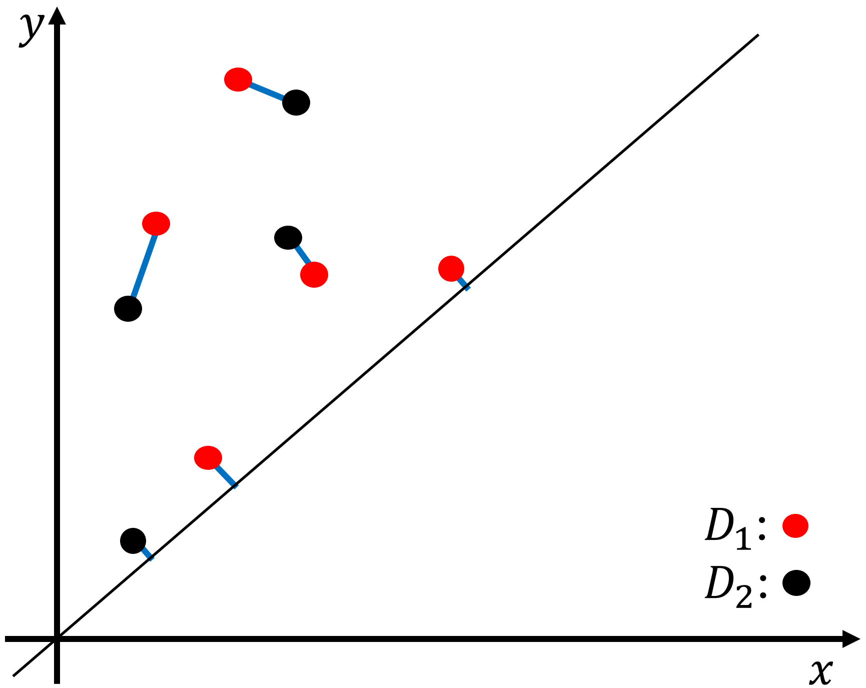

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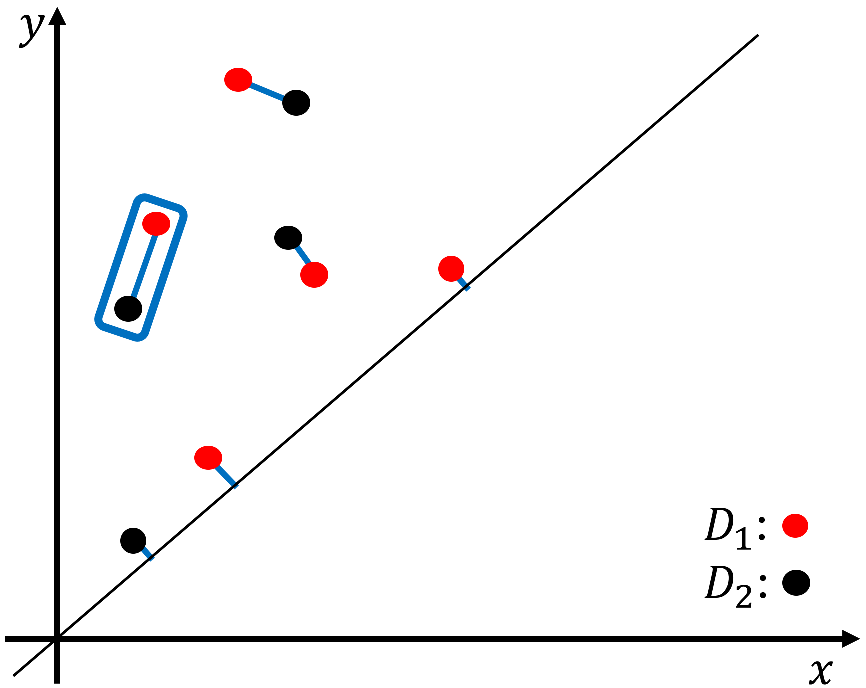

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The **bottleneck distance** is then defined as follows:

$$d_\text{B}(D_1, D_2) = \min_{\eta \text{ over all partial matchings}}\{cost(\eta)\}$$

aka. the minimum cost of all partial matchings

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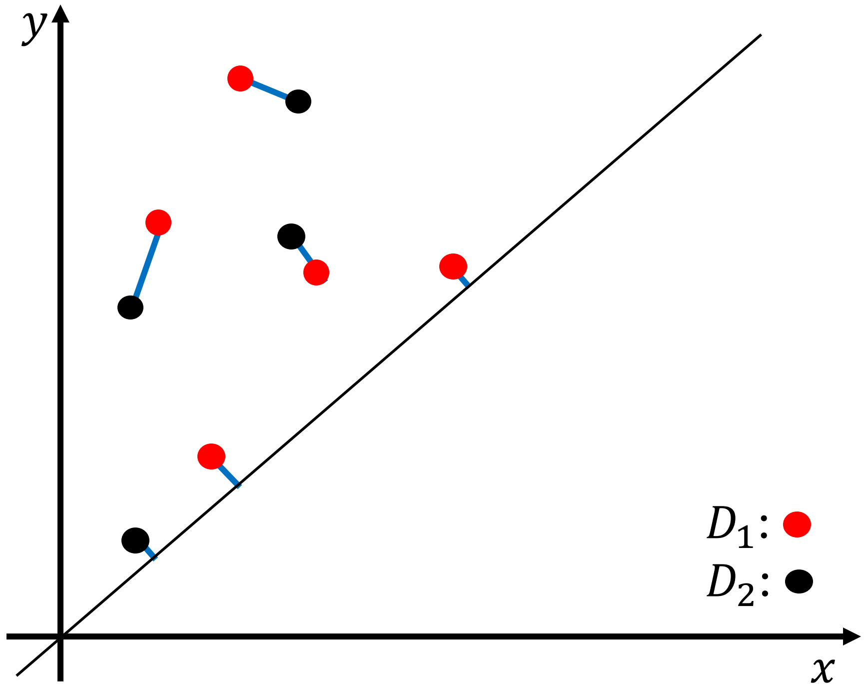
- So overall the two PDs are close

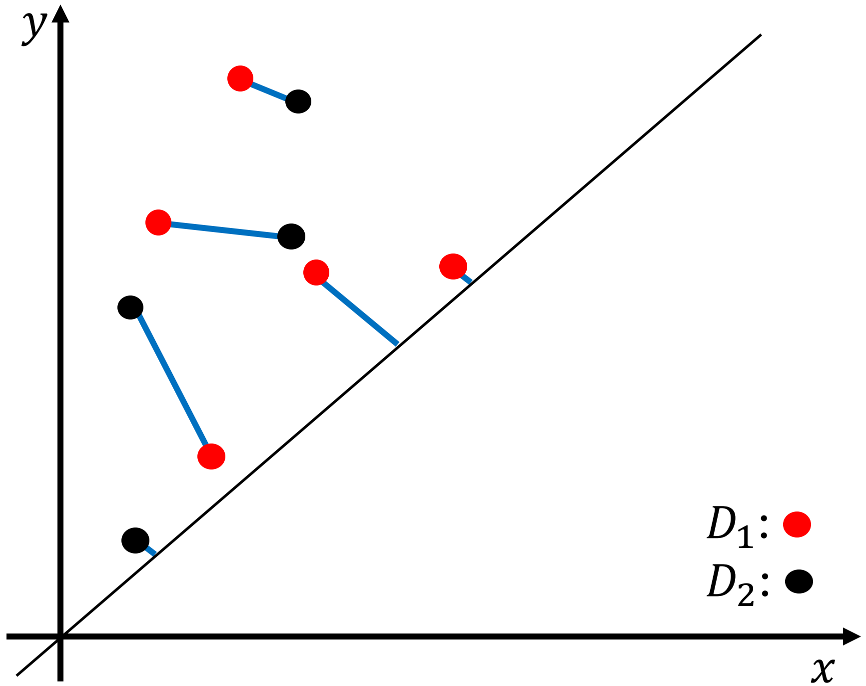
Distance for PDs

- The following is a "close" partial matching (with small cost) which achieves the bottleneck distance between D_1, D_2 (aka. $d_{\mathrm{B}}(D_1, D_2) = cost(\eta)$)



Distance for PDs

- The following is a partial matching where the max distance between matched points is high



Stability Theorem for Persistent Homology

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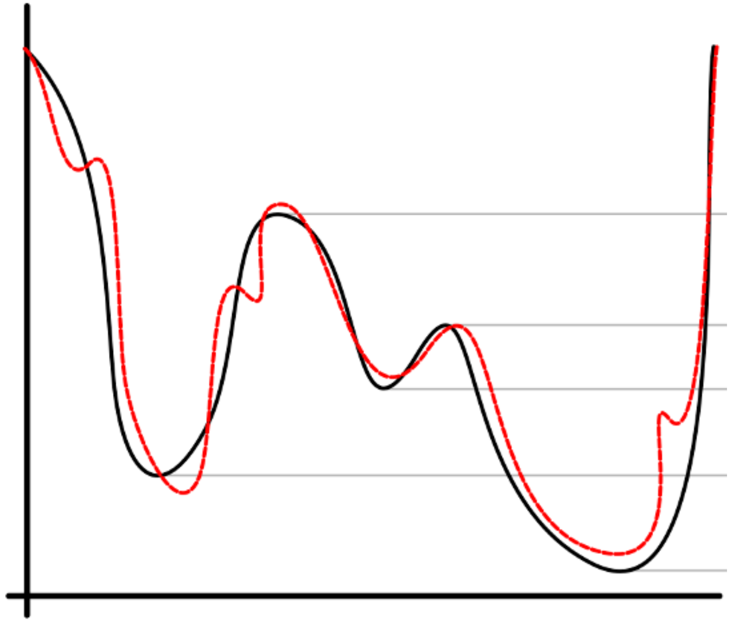

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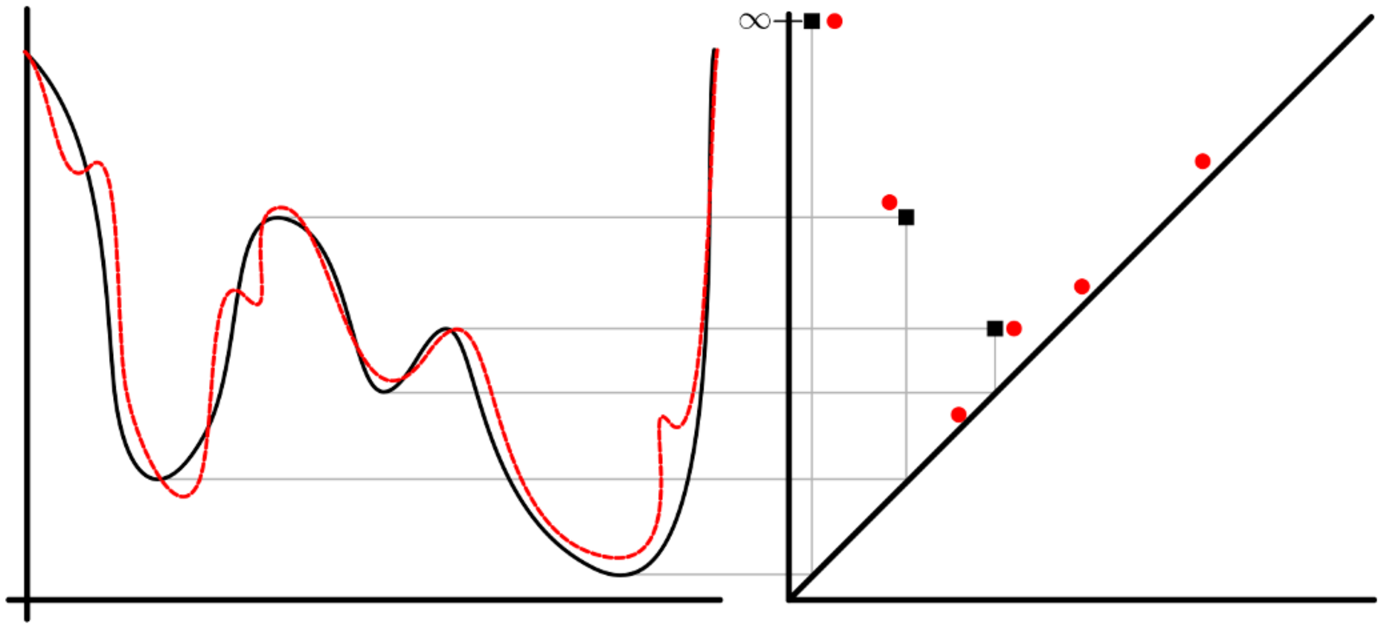

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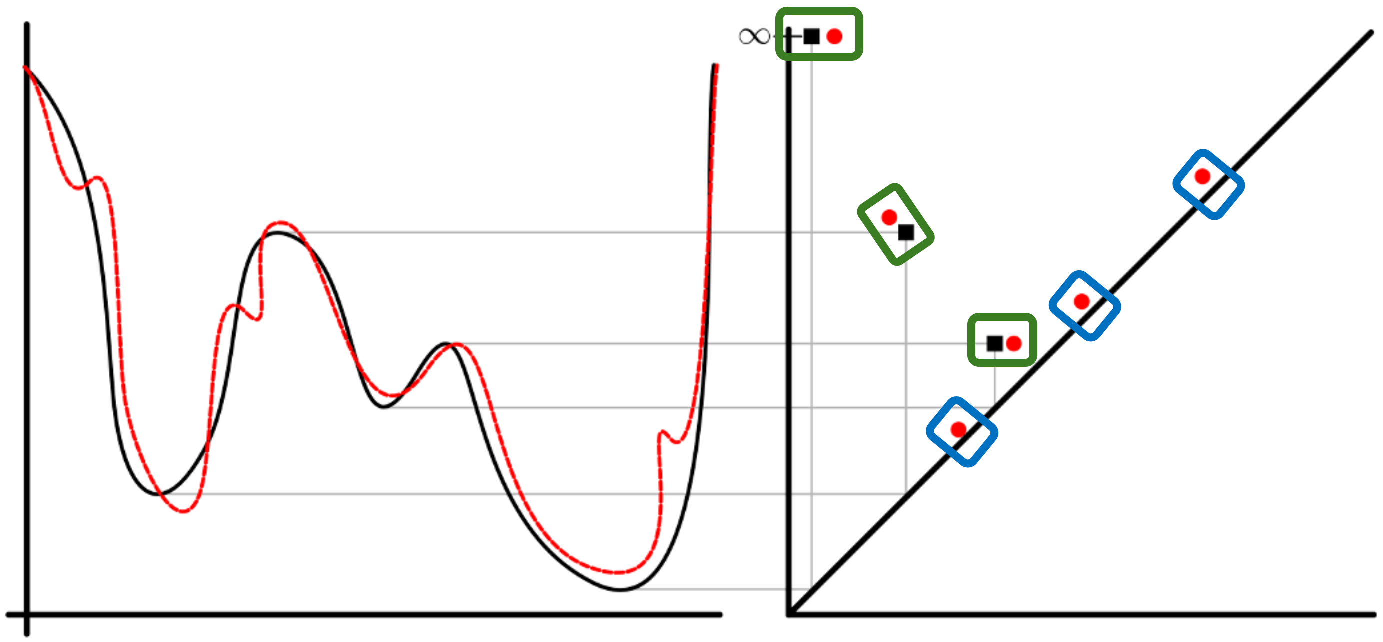

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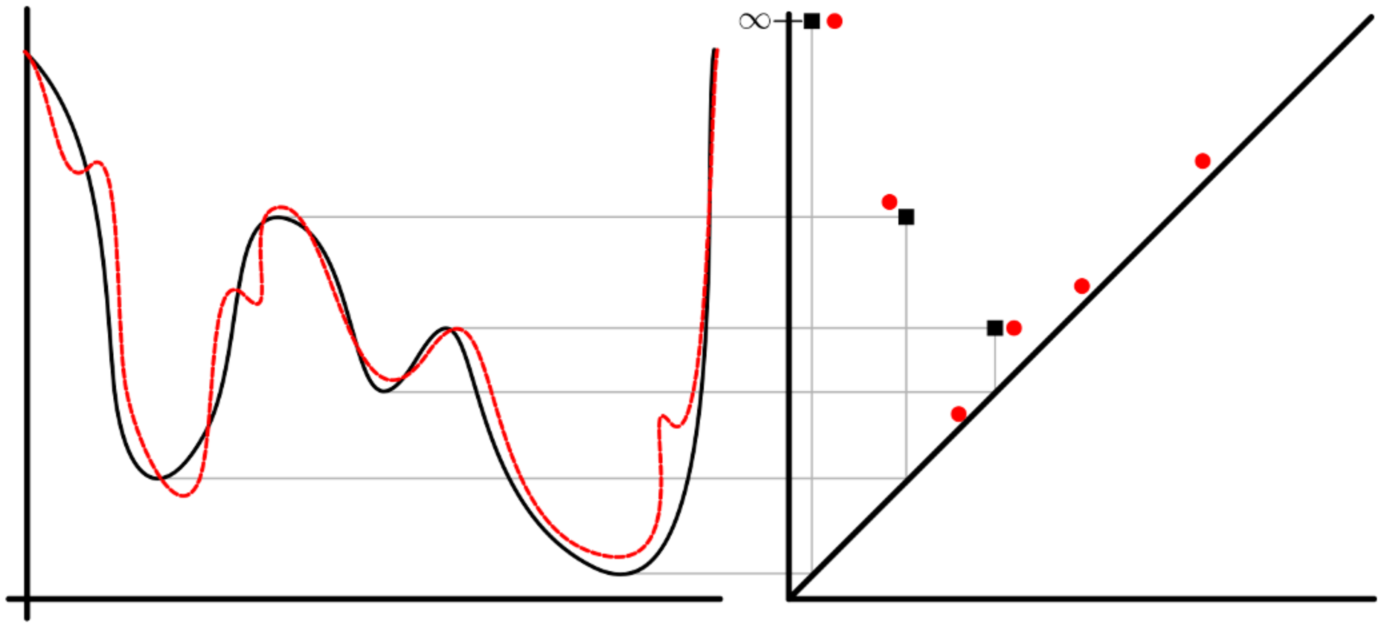

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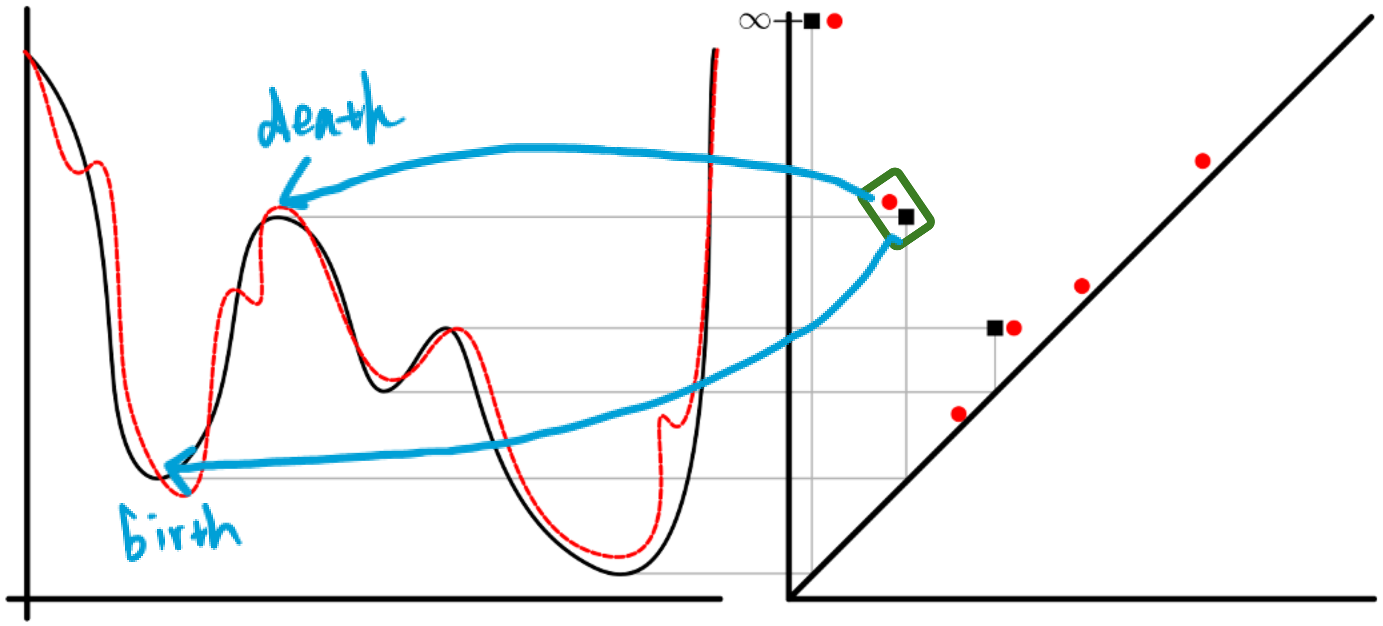

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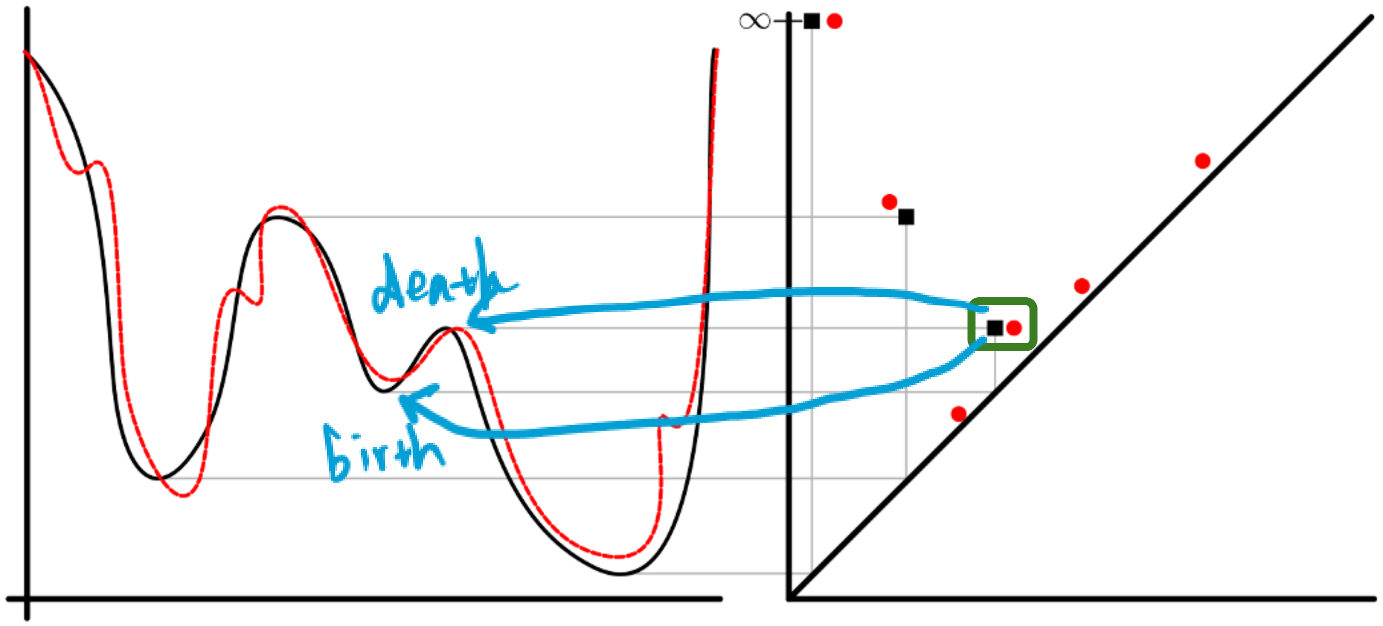

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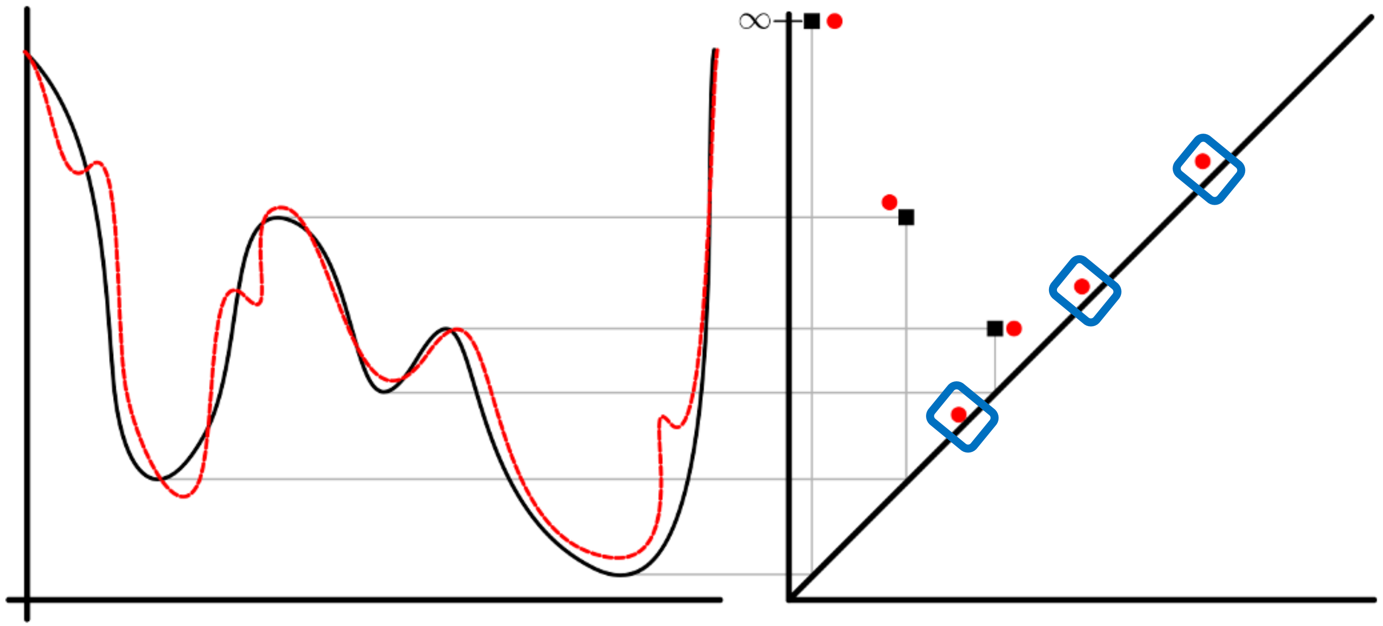

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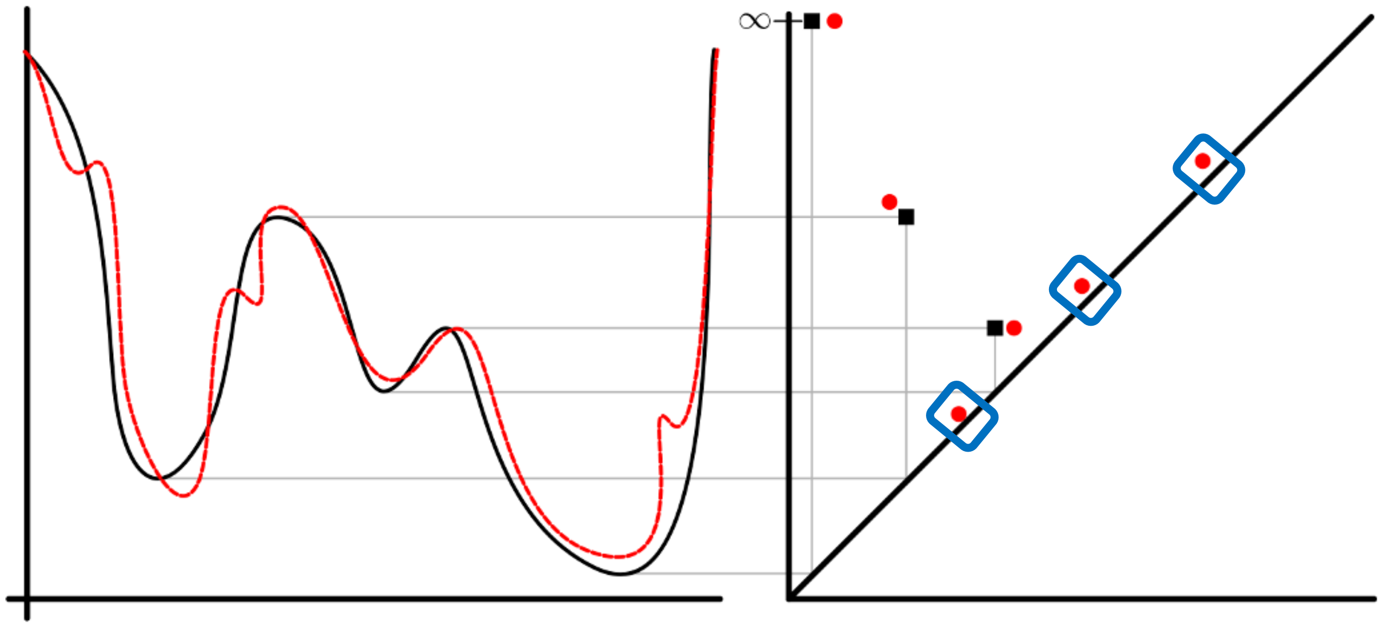

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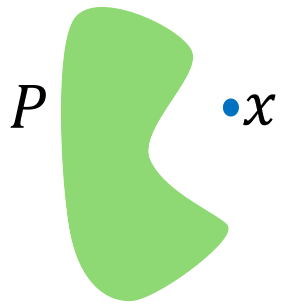

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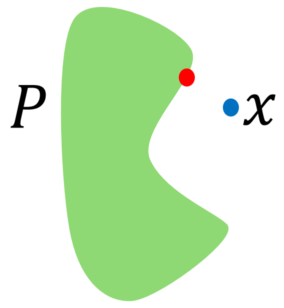

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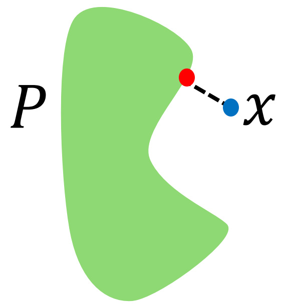

Distance field

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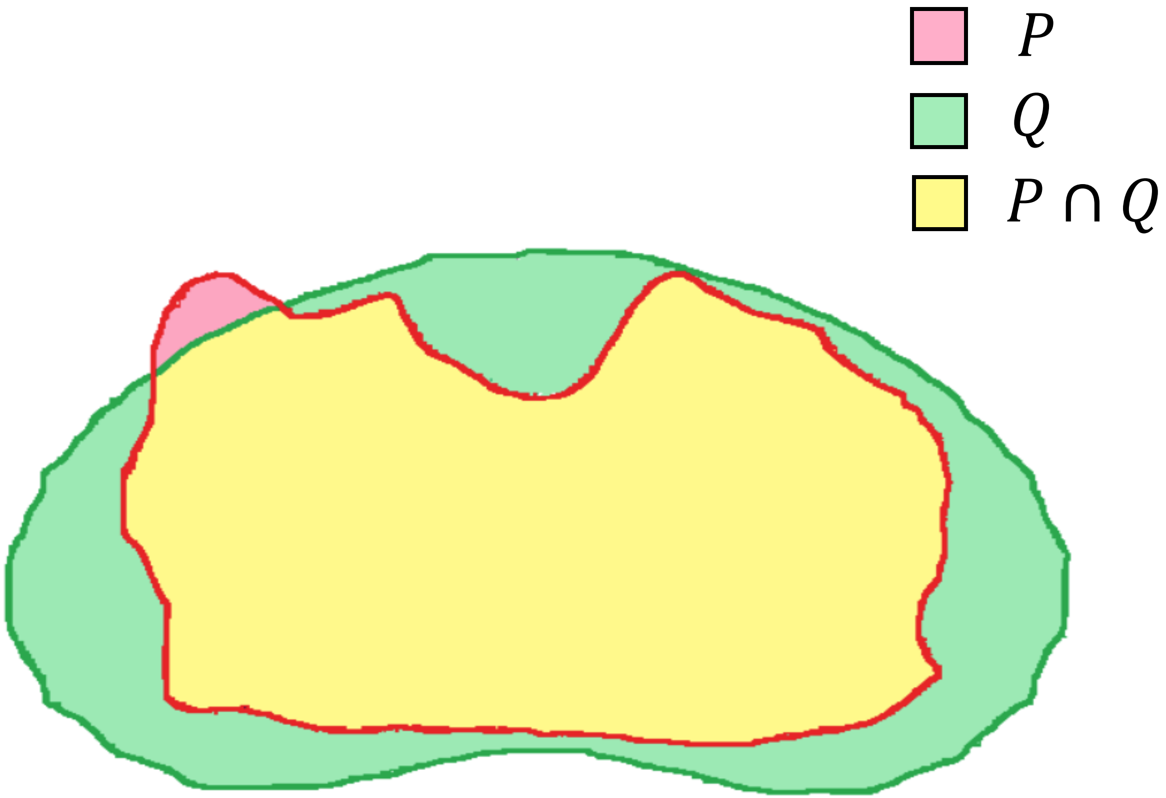

Img source: Karimi & Salcudean. Reducing the Hausdorff Distance in Medical Image Segmentation With Convolutional Neural Networks

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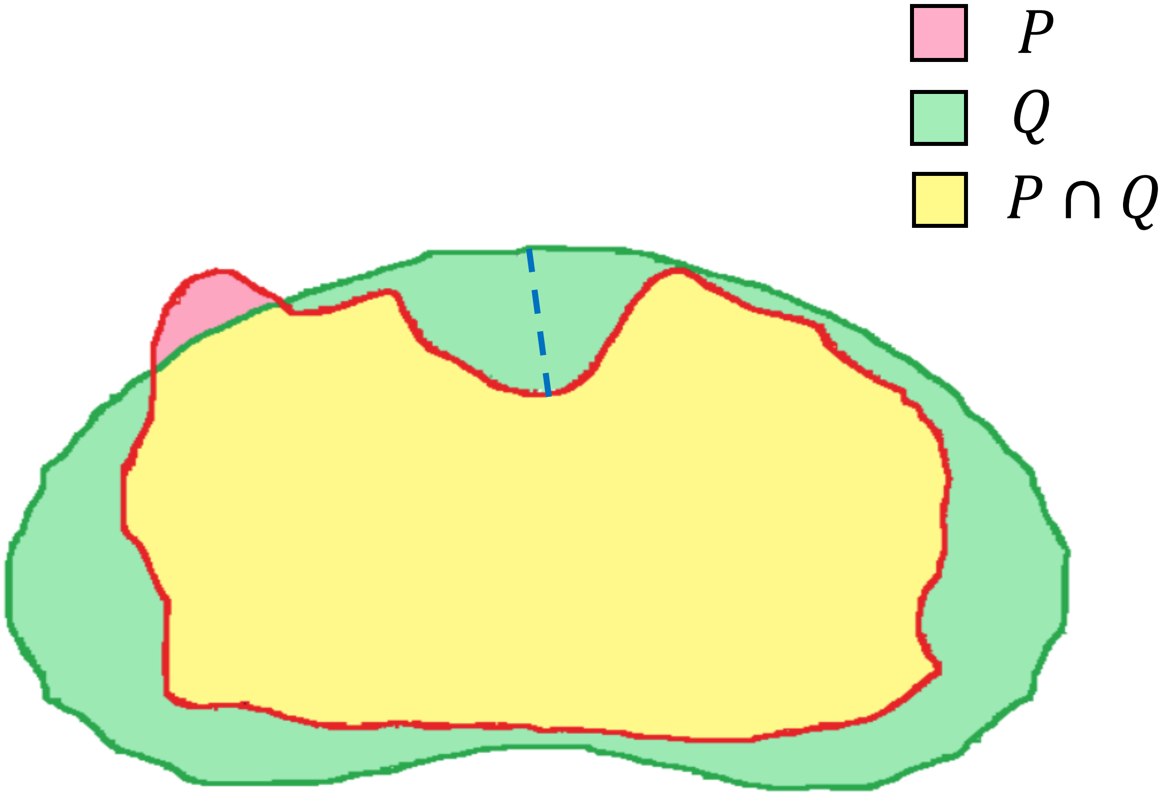

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- Based on the definition of Hausdorff distance, we have

$$d(p,q) = d(p,Q) \leq d_H(P,Q)$$

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- **Proof**: First of all

$$PD_{\check{C}ech}(P) = PD(f_P) \text{ and } PD_{\check{C}ech}(Q) = PD\left(f_Q\right)$$

based on previous corollary

Stability for point clouds

- Define $PD_{\check{C}ech}(P) \coloneqq PD(\mathcal{C}(P))$ for a point cloud P and recall that $\mathcal{C}(P)$ is the Čech filtration of P
- **Stability Theorem for Point Clouds**: One has

$$d_{\mathrm{B}}(PD_{\check{C}ech}(P), PD_{\check{C}ech}(Q)) \le d_H(P, Q)$$

- **Proof**: First of all

$$PD_{\check{C}ech}(P) = PD(f_P) \text{ and } PD_{\check{C}ech}(Q) = PD\big(f_Q\big)$$

based on previous corollary

- We then have

$$d_{\mathrm{B}}(PD_{\check{C}ech}(P), PD_{\check{C}ech}(Q)) = d_{\mathrm{B}}(PD(f_P), PD(f_Q)) \le \| f_P - f_Q \|_\infty \le d_H(P, Q)$$

where the middle inequality is by previous stability theorem

Stability for point clouds

- Define $PD_{Rips}(P) \coloneqq PD(\mathcal{R}(P))$ for a point cloud P and recall that $\mathcal{R}(P)$ is the Rips filtration of P
- **Stability Theorem for Point Clouds (Vietoris-Rips):** One has

$$d_{\mathrm{B}}(PD_{Rips}(P), PD_{Rips}(Q)) \leq d_H(P, Q)$$

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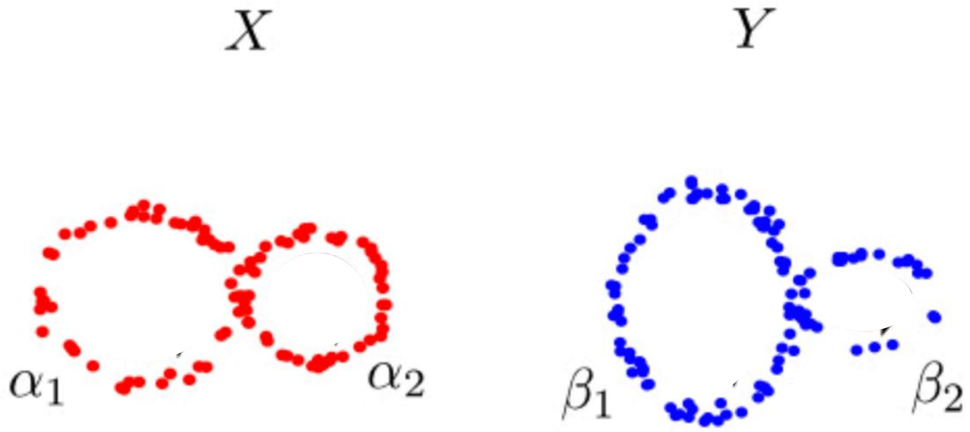
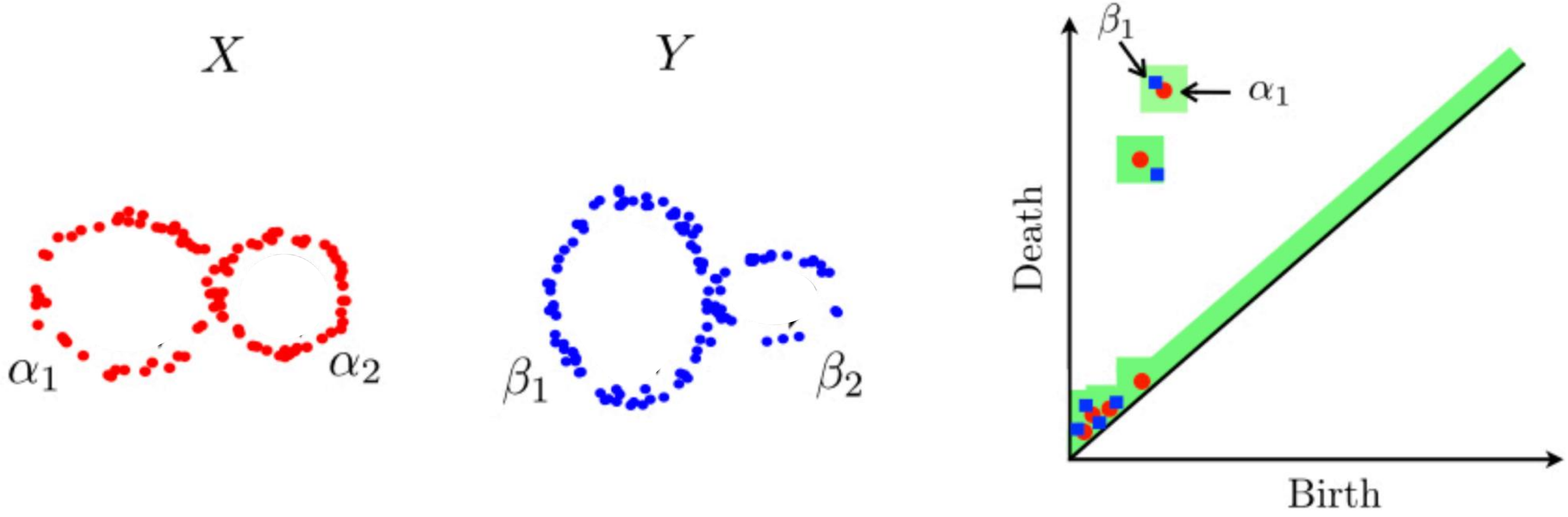
- Proof of this needs the more advanced notion of "interleaving" stability and is beyond scope

Stability for point clouds



Img source: Kusano, Fukumizu, Hiraoka. Kernel Method for Persistence Diagrams via Kernel Embedding and Weight Factor

Stability for point clouds



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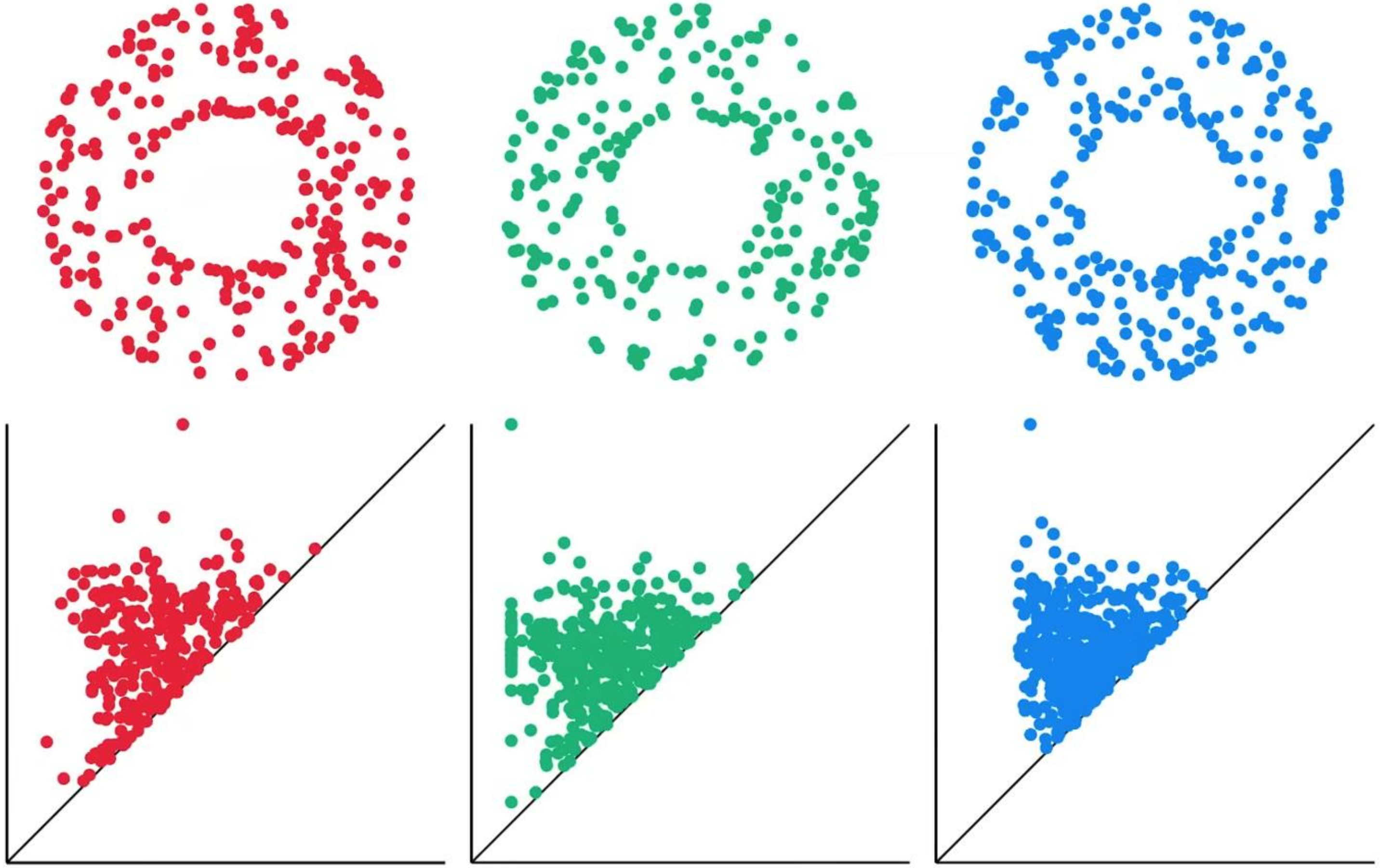

Figure from: Bastian Rieck: Topological Data Analysis for Machine Learning III: Topological Descriptors & How to Use Them (https://www.youtube.com/watch?app=desktop&v=7i1kabhl5IU)

Similarity of Čech and Rips PD

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- For a point cloud P, define $PD^{\log}_{\check{C}ech}(P)$ as the PD derived from $PD_{\check{C}ech}(P)$ by taking the logarithm of the birth and death value for each point (with an arbitrary but fixed base)

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Similarity of Čech and Rips PD

- For a point cloud P, define $PD_{\check{C}ech}^{\log}(P)$ as the PD derived from $PD_{\check{C}ech}(P)$ by taking the logarithm of the birth and death value for each point (with an arbitrary but fixed base)
- Define $PD_{Rips}^{\log}(P)$ similarly
- **Theorem**: One has

$$d_{\mathrm{B}}(PD_{\check{C}ech}^{\log}(P), PD_{Rips}^{\log}(P)) \leq \log 2$$

Similarity of Čech and Rips PD

- For a point cloud P, define $PD^{\log}_{Čech}(P)$ as the PD derived from $PD_{Čech}(P)$ by taking the logarithm of the birth and death value for each point (with an arbitrary but fixed base)
- Define $PD^{\log}_{Rips}(P)$ similarly
- **Theorem**: One has

$$d_{\mathrm{B}}(PD^{\log}_{Čech}(P), PD^{\log}_{Rips}(P)) \leq \log 2$$

- The theorem follows from the previous "interleaving" between Čech and Rips complexes (but details omitted)

$$\mathbb{C}^{\alpha}(P) \subseteq \mathbb{VR}^{\alpha}(P) \subseteq \mathbb{C}^{2\alpha}(P)$$

Computing bottleneck distance

- https://github.com/nihell/tutorialathon/blob/master/BottleneckTutorial.ipynb
- https://www.youtube.com/watch?v=4WswT9snTjc

A practical use of persistence barcode

- A practical use of drawing a PD as a barcode is that barcode provides a way to “visualize” the Betti number across the different range (value α)

Image source: Bobrowski O, Skraba P. A universal null-distribution for topological data analysis.

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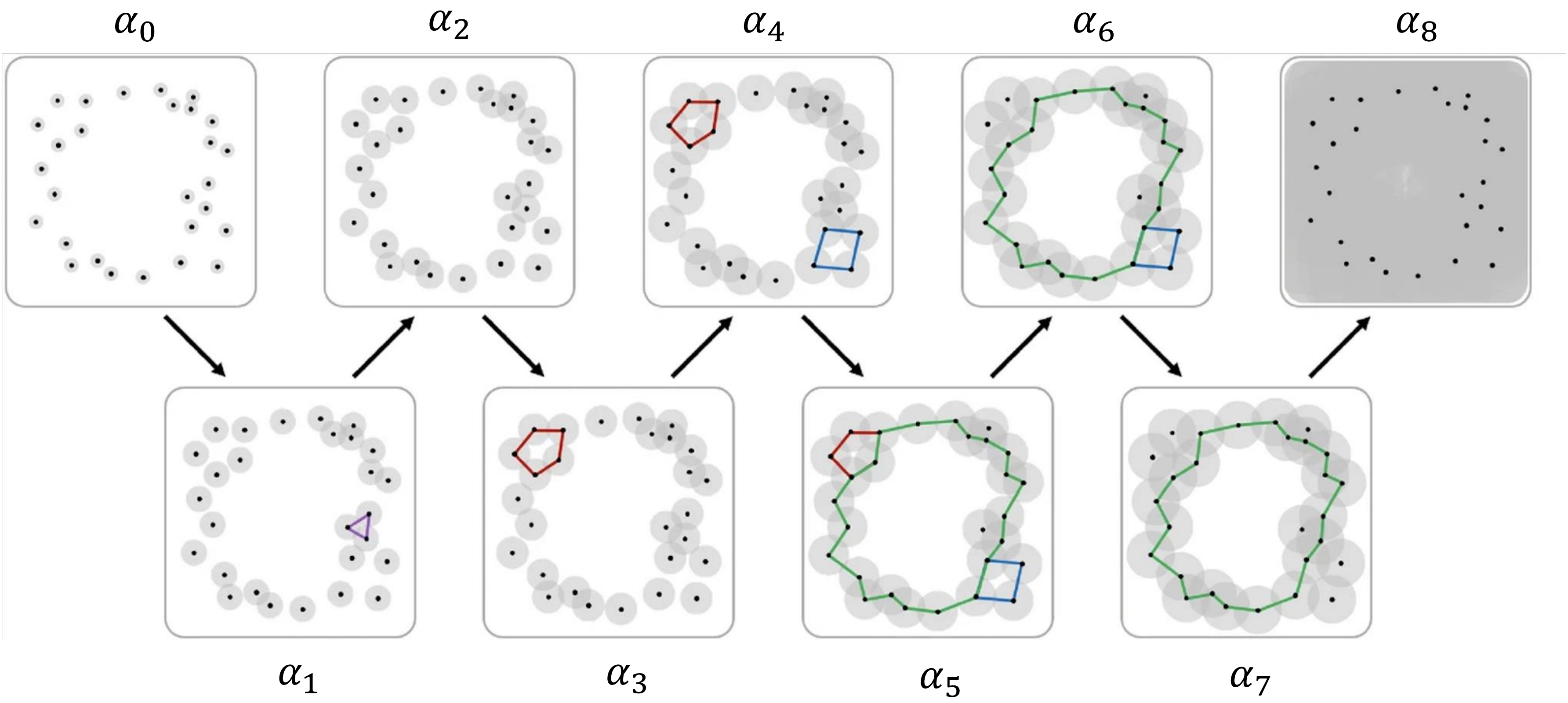
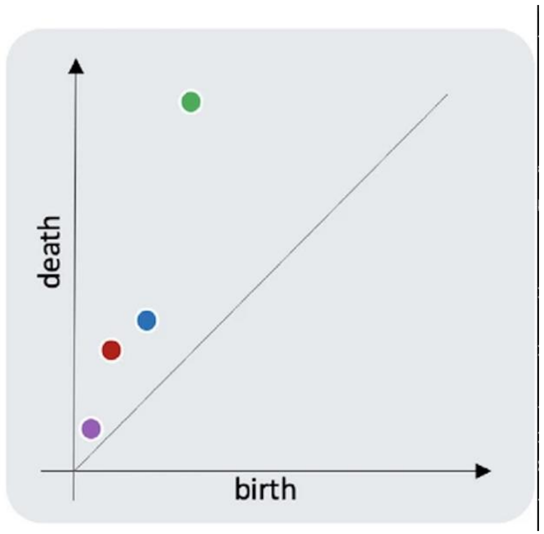
- A practical use of drawing a PD as a barcode is that barcode provides a way to "visualize" the Betti number across the different range (value α)
- For the previous filtration on a point cloud and its 1d PD





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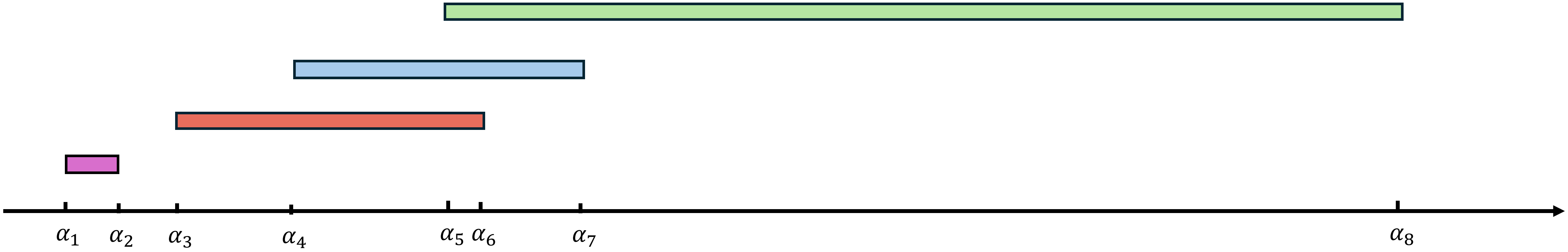

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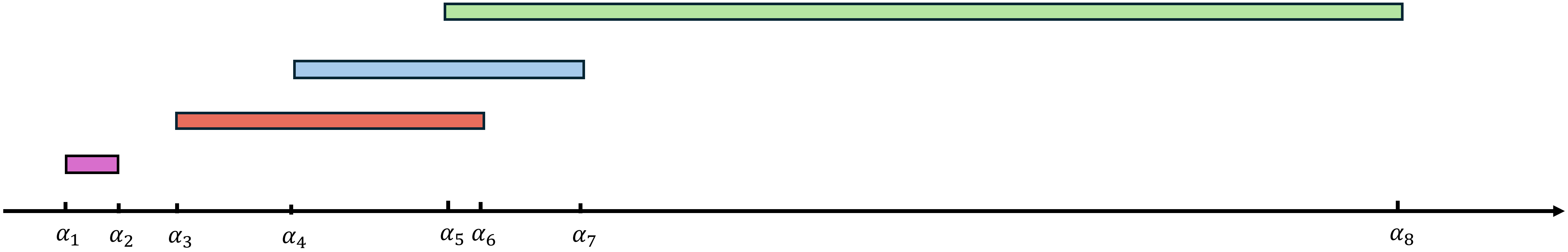
- The following is its 1d barcode
- Observe, if you count the number of intervals (bars) containing a certain value, then it gives you the 1st Betti for the complex corresponding to the range (value α) in the filtration



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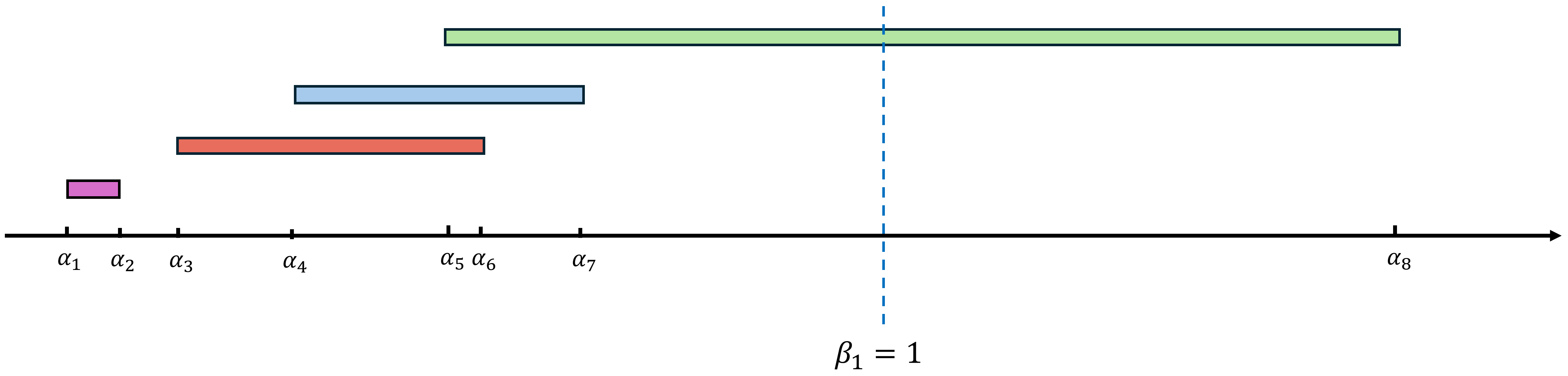
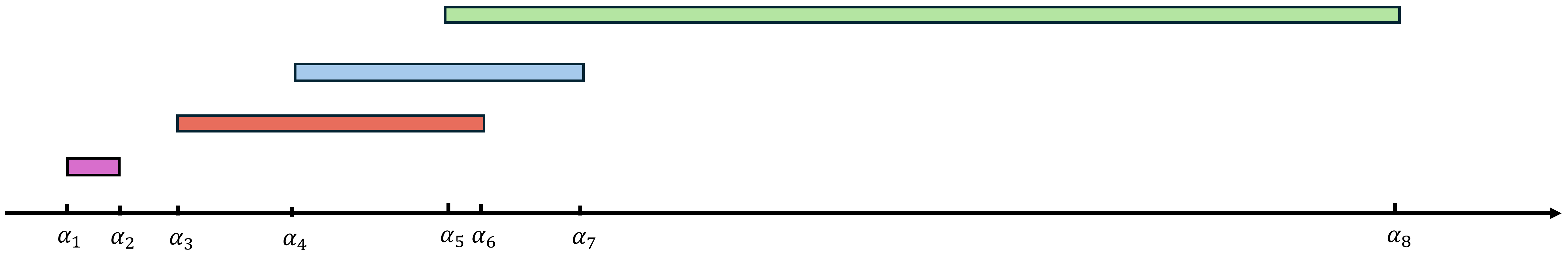


A practical use of persistence barcode

- More powerfully, if you look at the range of the values where the Betti number stays the same, and take the longest range, that would give you the most probably inference of the Betti number



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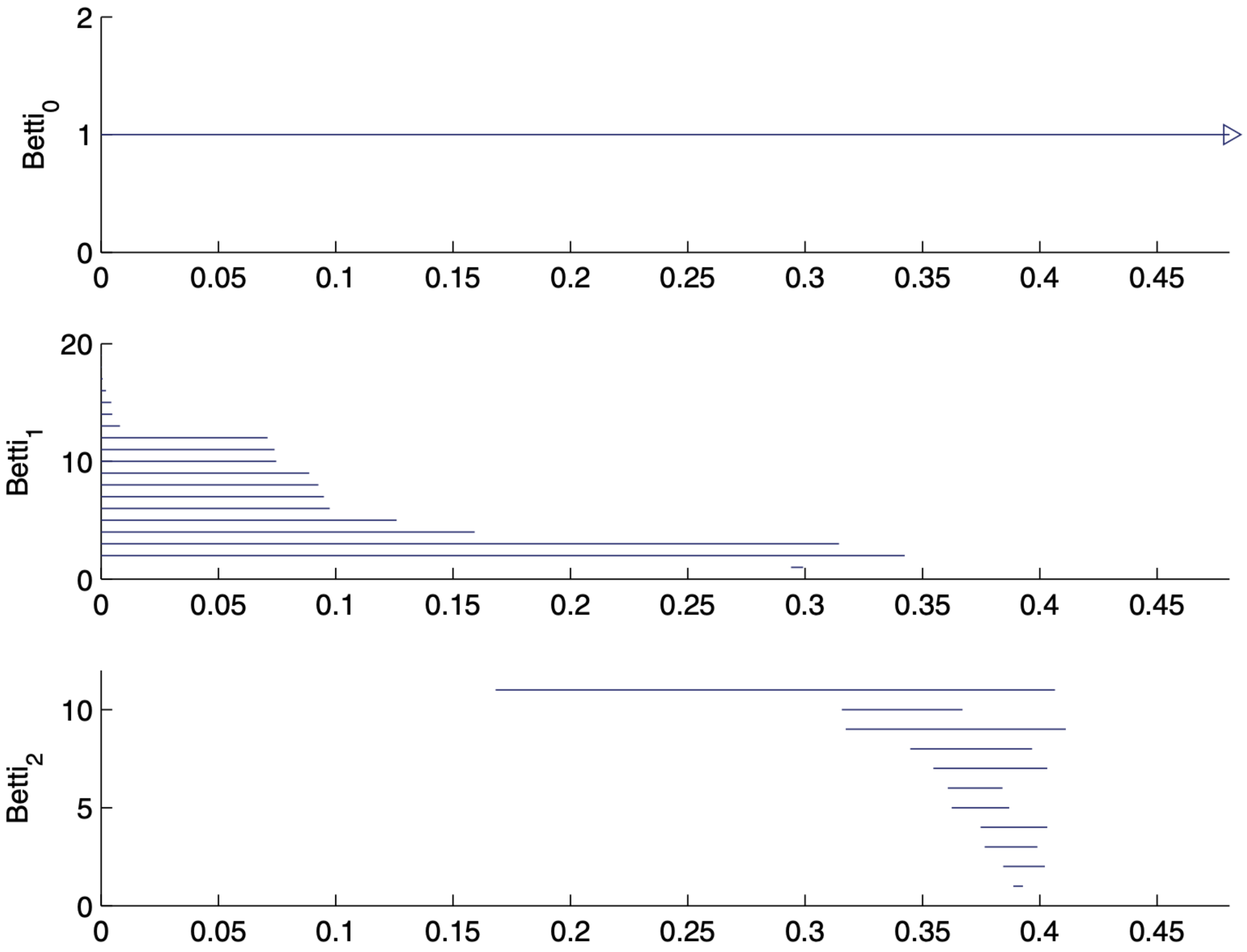
- More powerfully, if you look at the range of the values where the Betti number stays the same, and take the longest range, that would give you the most probably inference of the Betti number
- So most probably, for the previous point cloud, $\beta_1 = 1$ (complying out intuitions)

Another example



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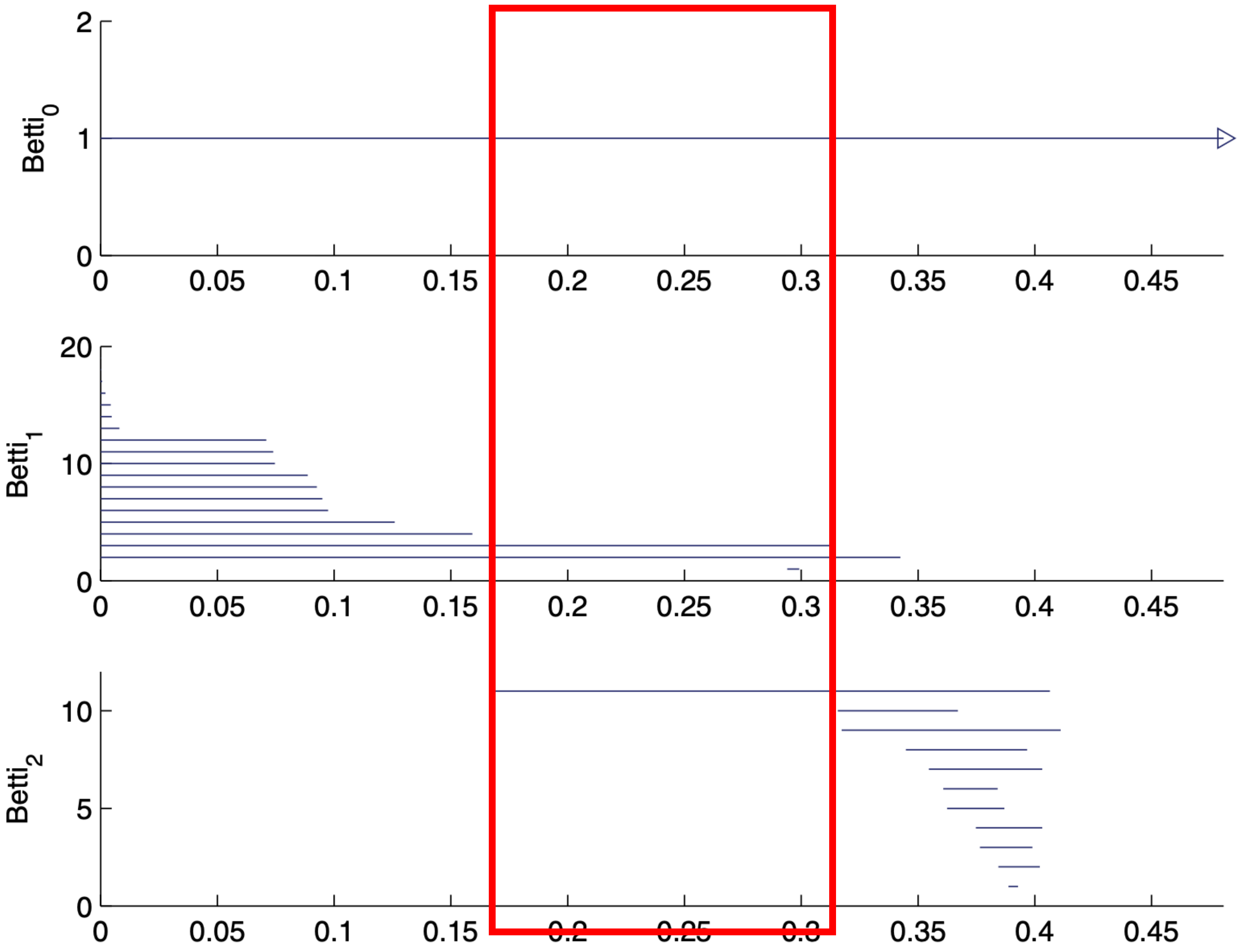


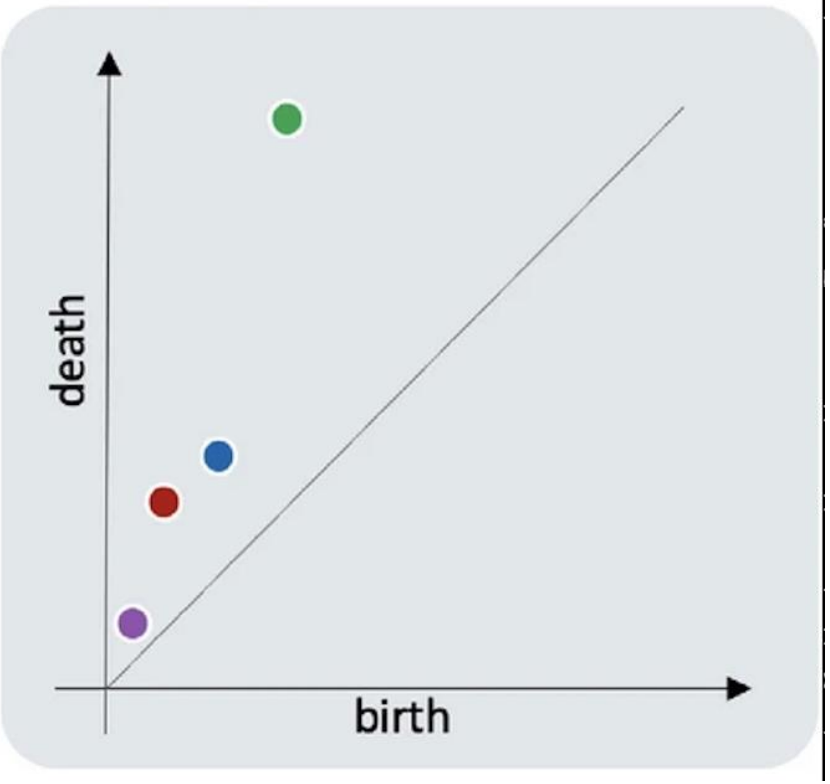
Counting Betti number using PD

- Now we know that taking the intersection of a vertical line with the intervals in the barcode gives you the Betti number at the value of the vertical line, what about PD? Can we do similar things in PD?

Image source: Bobrowski O, Skraba P. A universal null-distribution for topological data analysis.

Counting Betti number using PD

- It turns out that, the number of intervals intersecting a value α as manifested on the PD is the number of pointing in the upper-left quadrant of the point (α, α) on the diagonal



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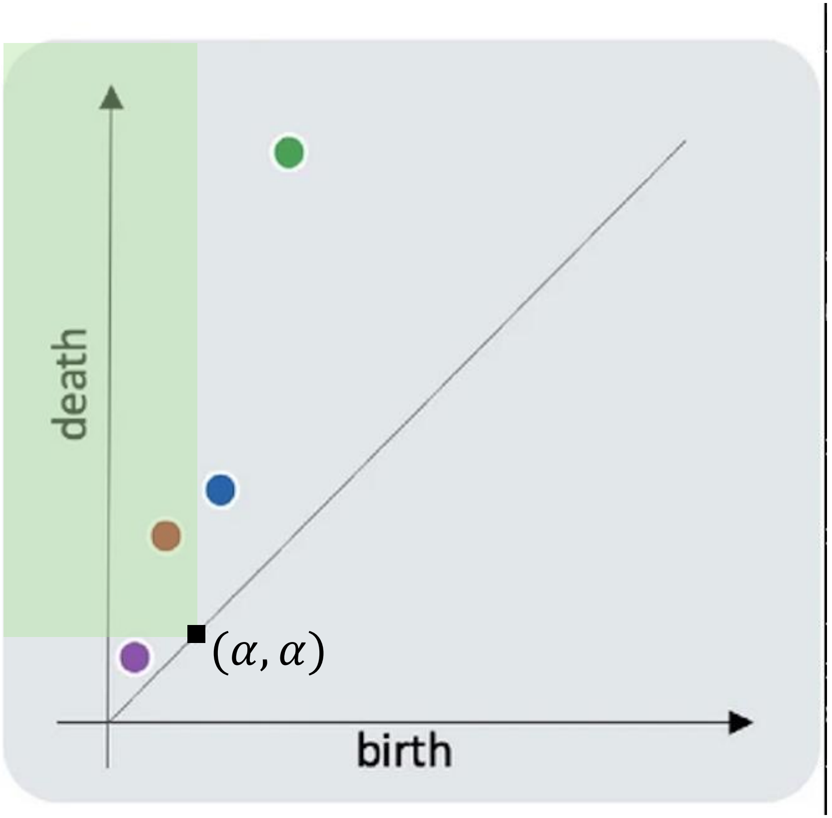

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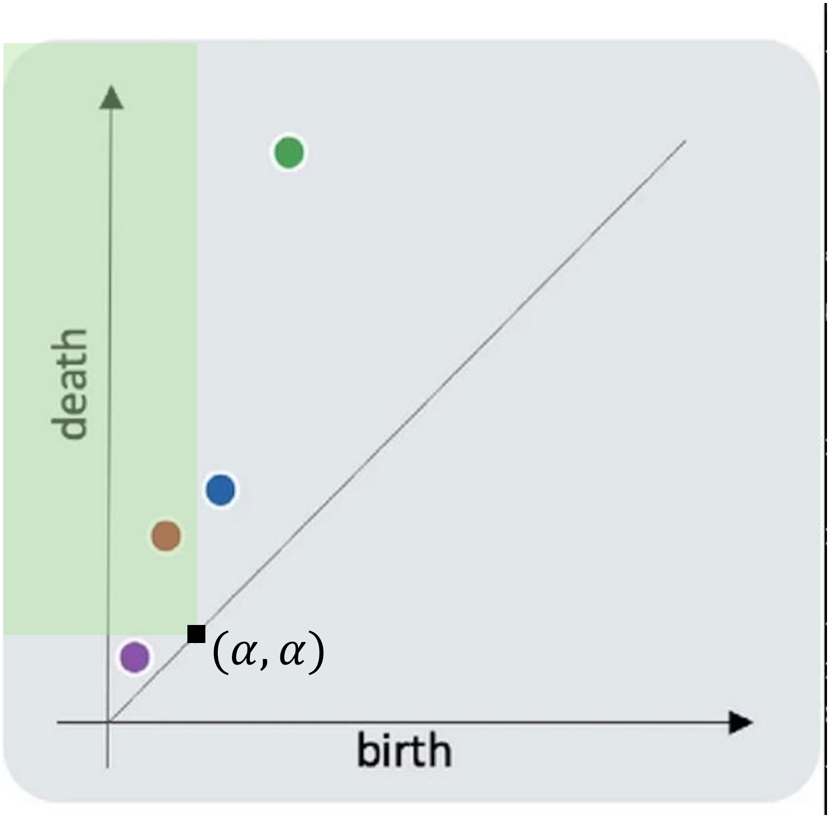
On barcode

α_1
α_2
α_3
α_4
α_5
α_6
α_7
α_8
$\beta_1 = 1$

On PD

- $\alpha_3 < \alpha < \alpha_4$



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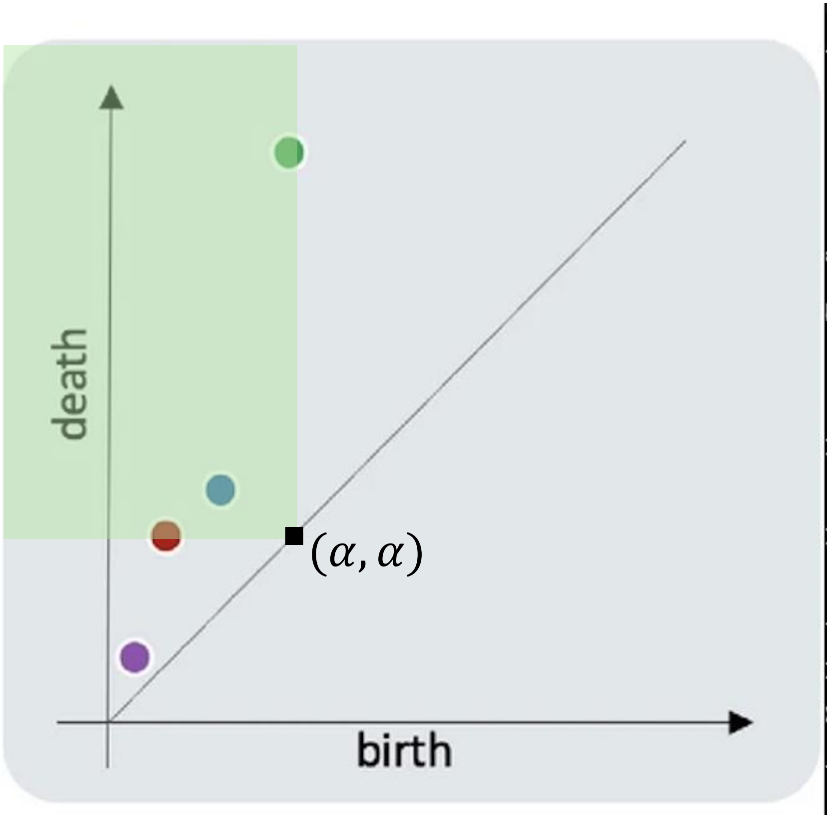
On Barcode

α_1
α_2
α_3
α_4
α_5
α_6
α_7
α_8
$\beta_1 = 3$

On PD

- $\alpha_5 < \alpha < \alpha_6$



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Length “cut-off” for intervals

- A typically belief in TDA is that people often think of features (intervals) with long lifespans as robust, important features, whereas a short lifespan may be an indication that the feature is less essential and may in fact be due to noise in data
- A commonly adopted approach in using PD/barcode for inference is to find a “cut-off” for the short intervals (noise) , and focus on the long intervals (actual features) only

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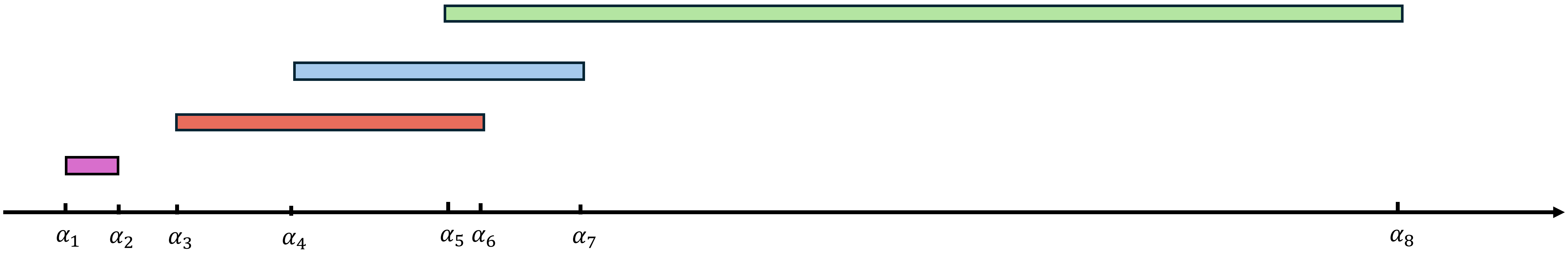

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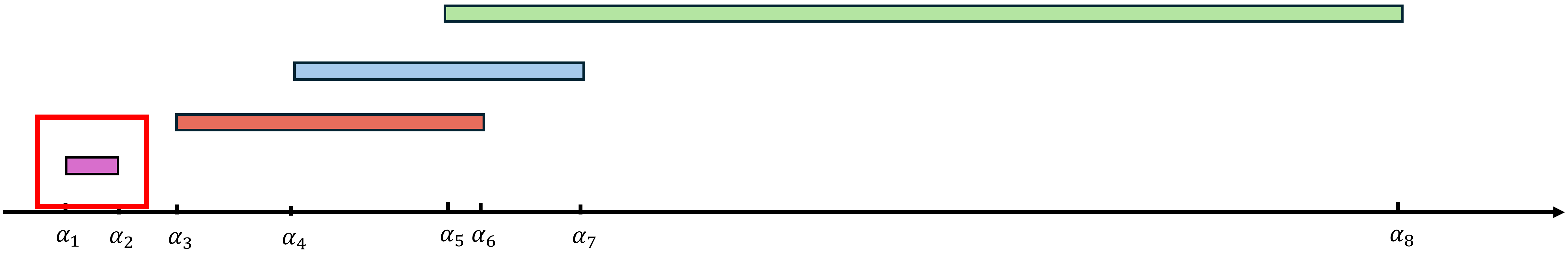

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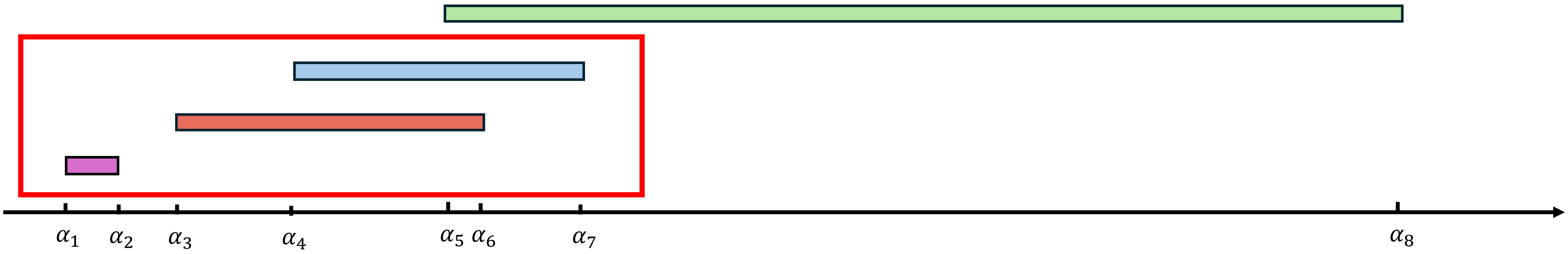

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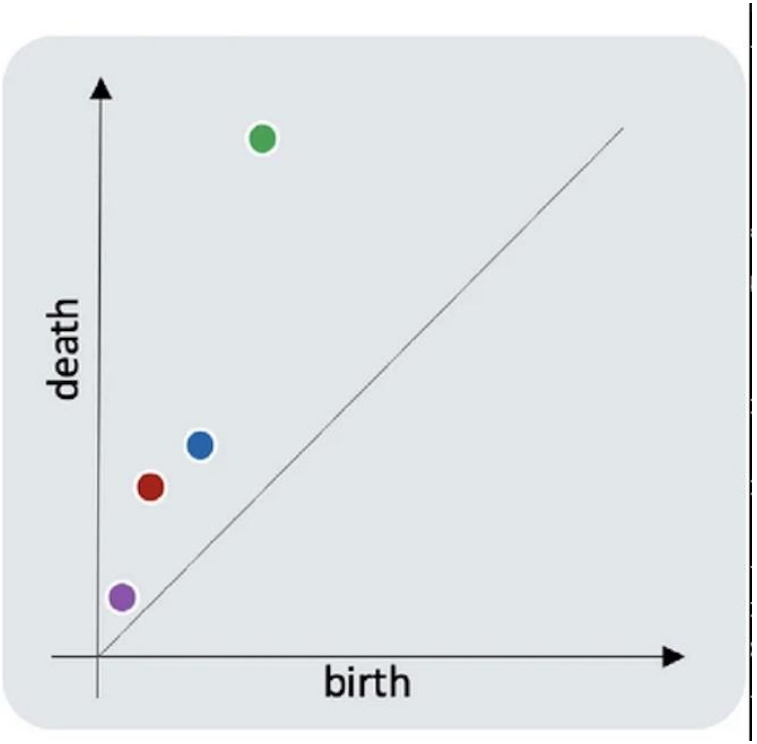

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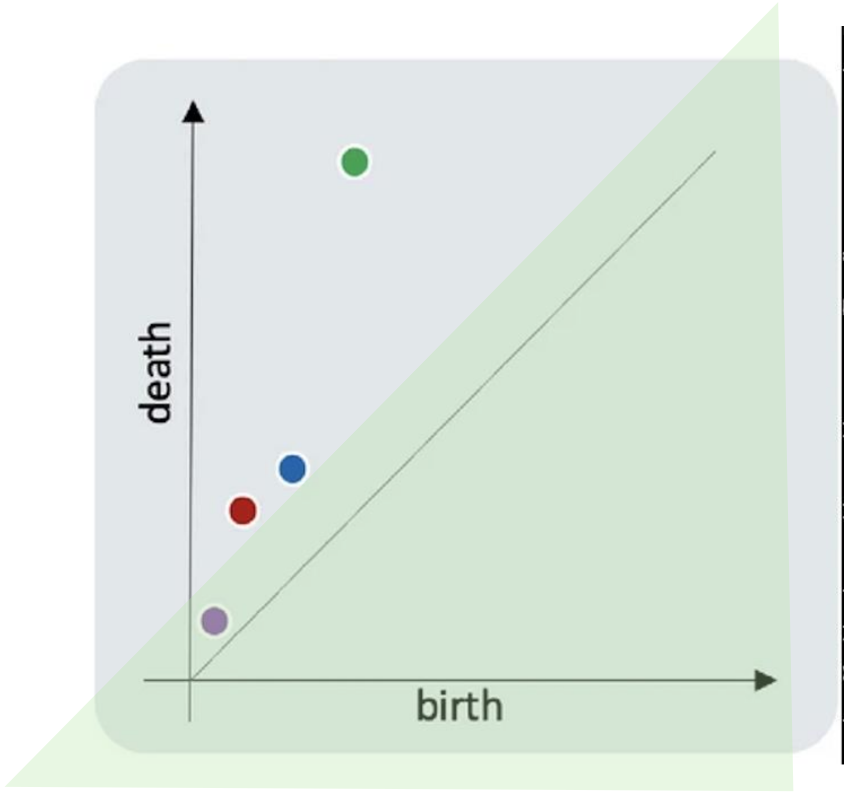

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