Computing Minimal Persistent Cycles: Polynomial and Hard Cases

Tamal K. Dey, **Tao Hou**, and Sayan Mandal

Department of Computer Science and Engineering
The Ohio State University

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Barcode/Persistence diagram

$$\mathcal{F} : \varnothing = K_0 \xhookrightarrow{\sigma_1} K_1 \xhookrightarrow{\sigma_2} \cdots \xhookrightarrow{\sigma_{m-1}} K_{m-1} \xhookrightarrow{\sigma_m} K_m = K$$

$$\Downarrow$$

Birth and death of d-th homological features: $\mathrm{D}_d(\mathcal{F})$

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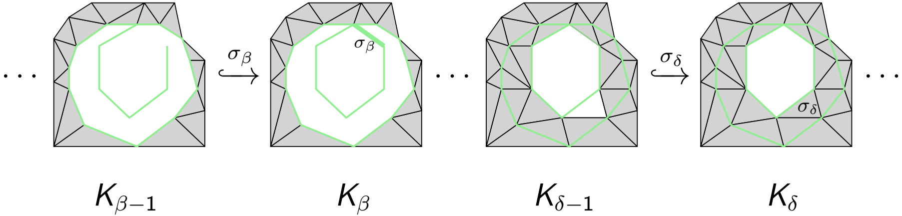
Birth and death of d-th homological features: $\mathrm{D}_d(\mathcal{F})$



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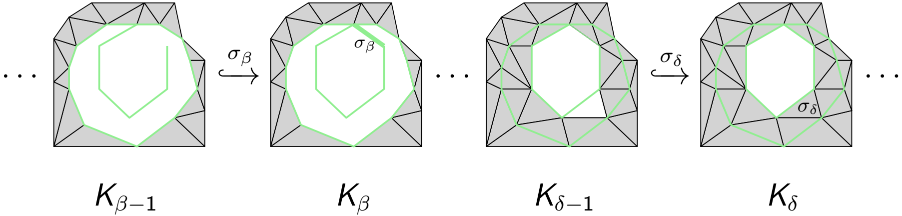
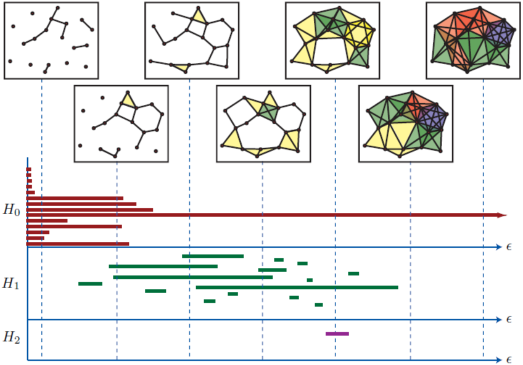


Therefore: An interval $[\beta, \delta) \in \mathrm{D}_1(\mathcal{F})$

Barcode/Persistence diagram



Two kinds of intervals

Finite interval: $[\beta, \delta)$ Infinite interval: $[\beta, +\infty)$

(Figure courtesy of [Ghrist, 2008])

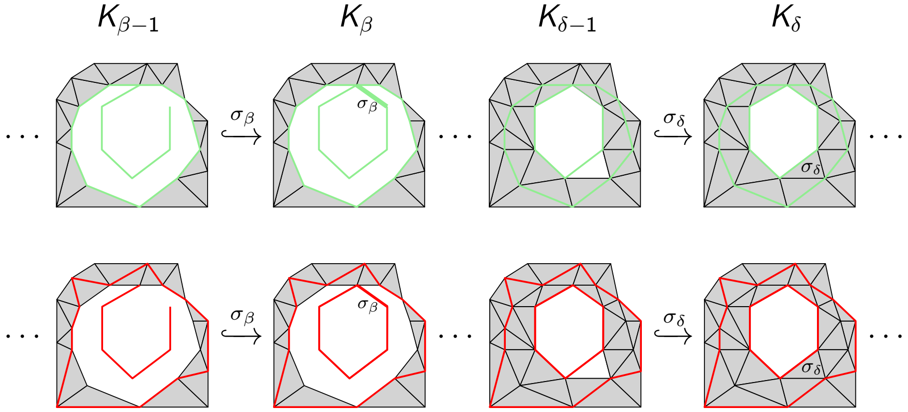
Definition

Definition (Persistent *d*-cycle)

For $[\beta, \delta) \in \mathrm{D}_d(\mathcal{F})$, it is a *d*-cycle ζ for $[\beta, \delta)$ *s.t.*

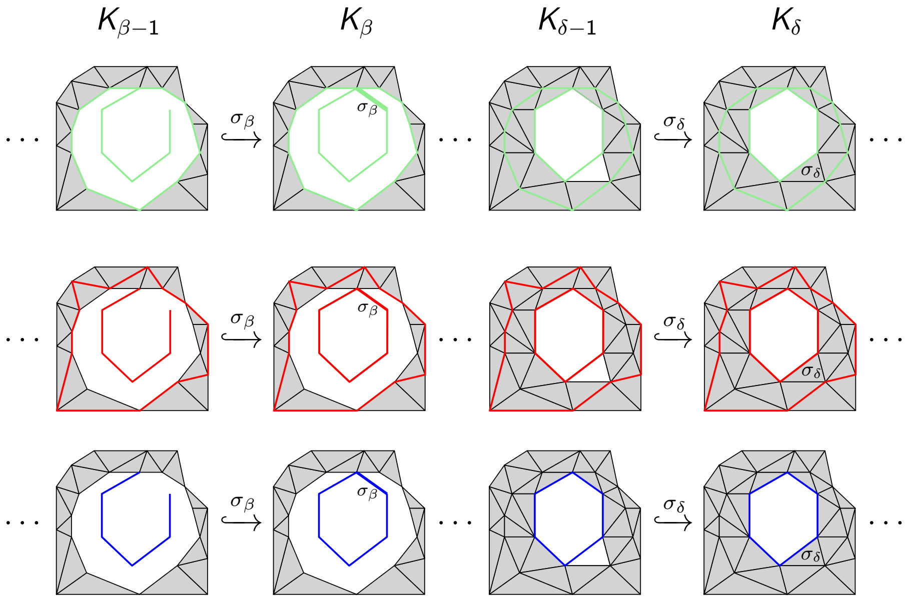
- $\delta = +\infty$ (infinite interval): ζ is a cycle in K_β containing σ_β
- $\delta \neq +\infty$ (finite interval): ζ is a cycle in K_β containing σ_β & ζ is not a boundary in $K_{\delta-1}$ but becomes boundary in K_δ

Definition



$K_{\beta-1}$
K_β
$K_{\delta-1}$
K_δ
σ_β
σ_β
σ_δ
σ_δ
σ_β
σ_β
σ_δ
σ_δ

Definition



Definition (Minimal persistent d-cycle)

- Each d-simplex has a (non-negative) weight
- Weight of d-cycle: Sum of weights of its d-simplices
- Persistent d-cycle for the interval with the minimal weight

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Problem (PCYC-FIN$_d$)

Given: Simplicial complex K, filtration $\mathcal{F}$, finite interval $[\beta, \delta) \in \mathsf{D}_d(\mathcal{F})$
Compute: A minimal persistent d-cycle for the interval

Problem (PCYC-INF$_d$)

Similar to PCYC-FIN$_d$, only interval $[\beta, +\infty)$ becomes infinite

Hardness results over dimension 1 [Dey et al., 2019]:

Problem	*d*	*Hardness*
PCYC-FIN$_1$	$= 1$	NP-hard
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New findings over general dimension:

Problem	*Restriction on K*	*d*	*Hardness*
PCYC-FIN$_d$	$-$	≥ 1	**NP-hard**
WPCYC-FIN$_d$	Weak $(d+1)$-pseudomanifold	≥ 1	**P**
PCYC-INF$_d$	$-$	$=1$	P
WPCYC-INF$_d$	Weak $(d+1)$-pseudomanifold	≥ 2	**NP-hard**
WEPCYC-INF$_d$	Weak $(d+1)$-pseudomanifold in $\mathbb{R}^{d+1}$	≥ 2	**P**

Definition

A simplicial complex K is a weak $(d+1)$-pseudomanifold if each d-simplex is face of no more than two $(d+1)$-simplices in K.

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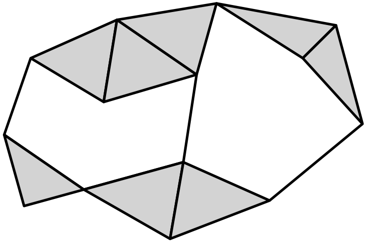
* Generalization of *pseudomanifold*

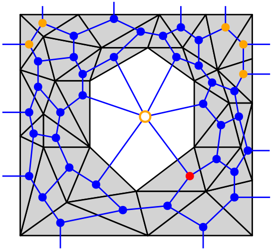


Minimal persistent d-cycles of finite intervals for weak $(d + 1)$-pseudomanifold

Duality between

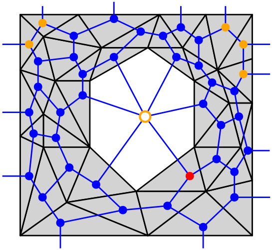
- persistent cycles of interval $[\beta, \delta)$
- s-t cuts on dual graph

Duality



Dual Graph

Duality

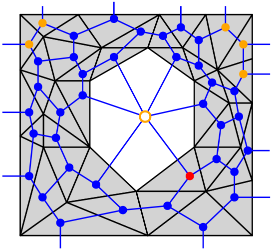


Dual Graph

Weight of edge:

- •Dual d-simplex in K_β: same weight
- •Dual d-simplex not in K_β: $+\infty$

Duality



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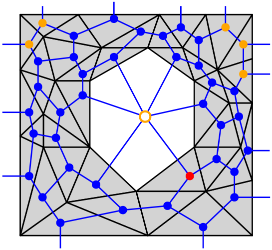
Src: Dual vertex of σ_δ

Duality



Dual Graph

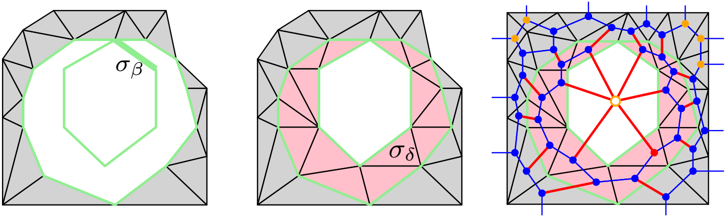
Weight of edge:

- •Dual d-simplex in K_β: same weight
- •Dual d-simplex not in K_β: $+\infty$

Src: Dual vertex of σ_δ

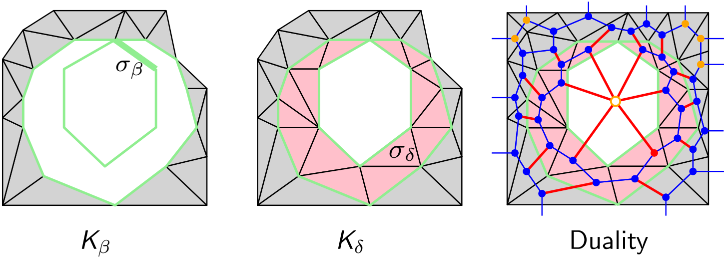
Sinks: Dual vertices of $(d+1)$-simplices **not** in K_δ $+$ the inf. vertex

Duality



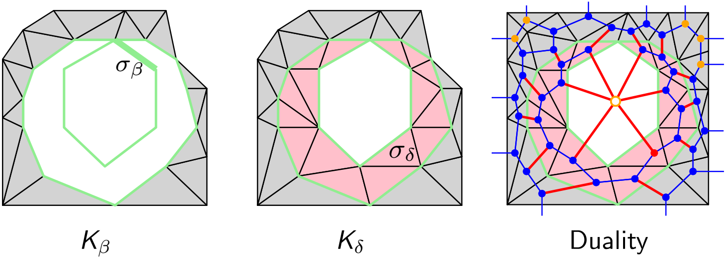
K_β K_δ Duality

Duality



Min-cut: (S, T)

Duality

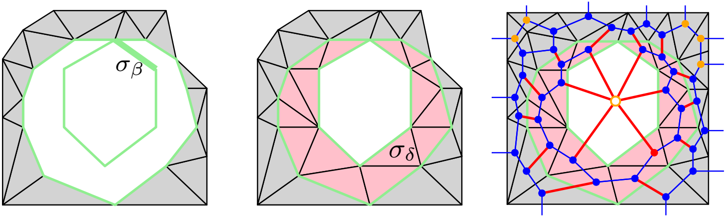


Min-cut: (S, T)

$$\Downarrow$$

Edges across (S, T)

Duality



K_β K_δ Duality

Min-cut: (S, T)

$\Downarrow$

Edges across (S, T)

$\Downarrow$

Dual d-chain: Minimal persistent d-cycle

Correctness of the Algorithm

Proposition

For any cut (S, T) *of* (G, s_1, s_2) *with finite weight, the d-chain* $\zeta = \theta^{-1}(\xi(S, T))$ *is a persistent d-cycle of* $[\beta, \delta)$ *and* $w(\zeta) = w(S, T)$

Proposition

For any persistent d-cycle ζ *of* $[\beta, \delta)$*, there exists a cut* (S, T) *of* (G, s_1, s_2) *such that* $w(S, T) \leq w(\zeta)$

Minimal persistent d-cycles of infinite intervals for weak $(d+1)$-pseudomanifold in $\mathbb{R}^{d+1}$

Aim: Find d-cycle with minimal weight containing σ_β in K_β

Similar duality as for finite interval

- No embedding assumption: NP-hard
- Needs some modification

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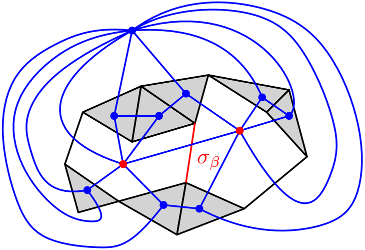


NP-hardness proof

The problems for which we prove the NP-hardness:

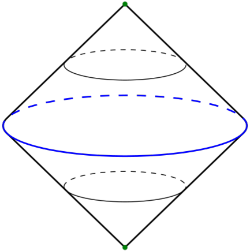
Problem	*Restriction on K*	*d*	*Hardness*
PCYC-FIN$_d$	$-$	≥ 1	**NP-hard**
WPCYC-INF$_d$	Weak $(d+1)$-pseudomanifold	≥ 2	**NP-hard**

NP-hardness proof

Suspension: Shifting dimension for reduction

Definition

$$\mathcal{S}K = \{\{\omega_1\},\{\omega_2\}\} \cup K \cup \left(\bigcup_{\sigma\in K}\{\sigma\cup\{\omega_1\},\sigma\cup\{\omega_2\}\}\right)$$



E.g., suspension of $\mathbb{S}^1 \Rightarrow \mathbb{S}^2$:

	$\widetilde{\mathrm{H}}_0$	$\widetilde{\mathrm{H}}_1$	$\widetilde{\mathrm{H}}_2$	$\widetilde{\mathrm{H}}_3$	$\cdots$
$\mathbb{S}^1$	0	$\mathbb{Z}_2$	0	0	$\cdots$
$\mathbb{S}^2$	0	0	$\mathbb{Z}_2$	0	$\cdots$

(Figure courtesy of Wikipedia)

Finite interval hardness

Proposition

PCYC-FIN$_{d-1}$ reduces to PCYC-FIN$_d$ for $d \geq 2$

Recall: PCYC-FIN$_1$ is NP-hard

Theorem

PCYC-FIN$_d$ is NP-hard for $d \geq 1$.

Infinite interval hardness

Theorem

WPCYC-INF$_2^+$ is NP-hard to approximate with any fixed ratio

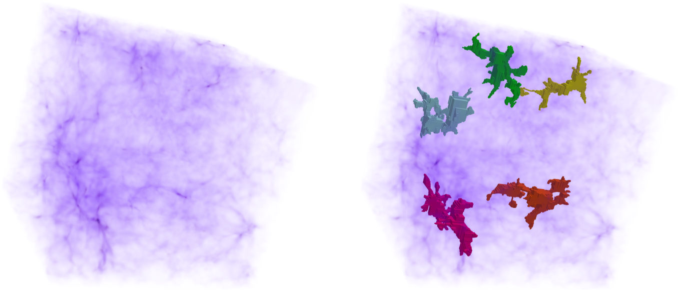
- WPCYC-INF$_2^+$: weights of the simplices are positive
- Reduce from *the nearest codeword problem*

Theorem

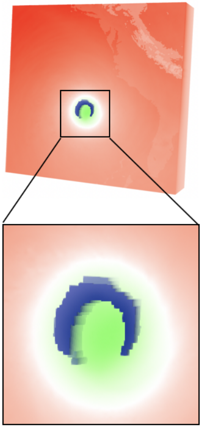
For $d \geq 2$, WPCYC-INF$_d^+$ is NP-hard to approximate with any fixed ratio

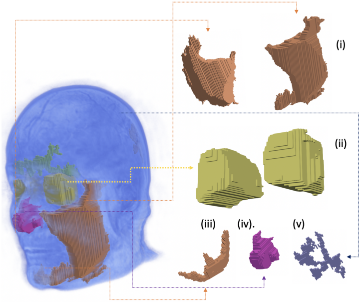
Shift the dimension by suspension

Cosmology



Hurricane/Medical MRI





References

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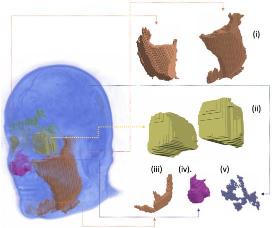
Thank You



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