Computing Minimal Persistent Cycles: Polynomial and Hard Cases

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Barcode/Persistence diagram

\[ \mathcal{F} : \emptyset = K_0 \overset{\sigma_1}{\leftarrow} K_1 \overset{\sigma_2}{\rightarrow} \cdots \overset{\sigma_{m-1}}{\rightarrow} K_{m-1} \overset{\sigma_m}{\leftarrow} K_m = K \]

\[ \Downarrow \]

Birth and death of \( d \)-th homological features: \( D_d(\mathcal{F}) \)
\[ \mathcal{F} : \emptyset = K_0 \xrightarrow{\sigma_1} K_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_{m-1}} K_{m-1} \xrightarrow{\sigma_m} K_m = K \]

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Birth and death of \( d \)-th homological features: \( D_d(\mathcal{F}) \)
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\[ \Downarrow \]

Birth and death of \( d \)-th homological features: \( D_d(\mathcal{F}) \)

Therefore: An interval \([\beta, \delta) \in D_1(\mathcal{F})\)
Barcode/Persistence diagram

Two kinds of intervals
Finite interval: $[\beta, \delta)$  Infinite interval: $[\beta, +\infty)$

(Figure courtesy of [Ghrist, 2008])
Definition (Persistent $d$-cycle)

For $[\beta, \delta) \in D_d(F)$, it is a $d$-cycle $\zeta$ for $[\beta, \delta)$ s.t.

- $\delta = +\infty$ (infinite interval): $\zeta$ is a cycle in $K_{\beta}$ containing $\sigma_{\beta}$
- $\delta \neq +\infty$ (finite interval): $\zeta$ is a cycle in $K_{\beta}$ containing $\sigma_{\beta}$ & $\zeta$ is not a boundary in $K_{\delta-1}$ but becomes boundary in $K_{\delta}$
Definition

\[ K_{\beta-1} \quad K_{\beta} \quad K_{\delta-1} \quad K_{\delta} \]

\[ \sigma_{\beta} \rightarrow \quad \sigma_{\beta} \rightarrow \quad \sigma_{\delta} \rightarrow \quad \sigma_{\delta} \rightarrow \]

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Definition

\( K_{\beta-1} \) \hspace{1cm} \( K_{\beta} \) \hspace{1cm} \( K_{\delta-1} \) \hspace{1cm} \( K_{\delta} \)

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Definition (Minimal persistent $d$-cycle)

- Each $d$-simplex has a (non-negative) weight
- Weight of $d$-cycle: Sum of weights of its $d$-simplices
- Persistent $d$-cycle for the interval with the minimal weight
Definition (Minimal persistent $d$-cycle)

- Each $d$-simplex has a (non-negative) weight
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Problem (PCYC-FIN$_d$)

Given: Simplicial complex $K$, filtration $\mathcal{F}$, finite interval $[\beta, \delta) \in D_d(\mathcal{F})$
Compute: A minimal persistent $d$-cycle for the interval

Problem (PCYC-INF$_d$)

Similar to PCYC-FIN$_d$, only interval $[\beta, +\infty)$ becomes infinite
Hardness results over dimension 1 [Dey et al., 2019]:

<table>
<thead>
<tr>
<th>Problem</th>
<th>$d$</th>
<th>Hardness</th>
</tr>
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<tbody>
<tr>
<td>PCYC-FIN$_1$</td>
<td>1</td>
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New findings over general dimension:

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<td>WEPCYC-INF$_d$</td>
<td>Weak $(d + 1)$-pseudomanifold in $\mathbb{R}^{d+1}$</td>
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Definition

A simplicial complex $K$ is a weak $(d + 1)$-pseudomanifold if each $d$-simplex is face of no more than two $(d + 1)$-simplices in $K$. 
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* Generalization of pseudomanifold
Minimal persistent $d$-cycles of finite intervals for weak $(d + 1)$-pseudomanifold

Duality between

- persistent cycles of interval $[\beta, \delta)$
- s-t cuts on dual graph
Duality

Dual Graph

Weight of edge:

- Dual $d$-simplex in $K_{\beta}$: same weight
- Dual $d$-simplex not in $K_{\beta}$: $+\infty$

Src: Dual vertex of $\sigma_{\delta}$

Sinks: Dual vertices of $(d+1)$-simplices not in $K_{\delta} +$ the inf. vertex
Duality

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Duality

\[ K_\beta \quad K_\delta \quad \text{Duality} \]
Min-cut: \((S, T)\)
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Edges across \((S, T)\)
Duality

Min-cut: \((S, T)\)

Edges across \((S, T)\)

Dual \(d\)-chain: Minimal persistent \(d\)-cycle
Correctness of the Algorithm

Proposition

For any cut \((S, T)\) of \((G, s_1, s_2)\) with finite weight, the \(d\)-chain 
\[ \zeta = \theta^{-1}(\xi(S, T)) \]
is a persistent \(d\)-cycle of \([\beta, \delta)\) and
\[ w(\zeta) = w(S, T) \]

Proposition

For any persistent \(d\)-cycle \(\zeta\) of \([\beta, \delta)\), there exists a cut \((S, T)\) of 
\((G, s_1, s_2)\) such that 
\[ w(S, T) \leq w(\zeta) \]
Minimal persistent $d$-cycles of infinite intervals for weak $(d + 1)$-pseudomanifold in $\mathbb{R}^{d+1}$

Aim: Find $d$-cycle with minimal weight containing $\sigma_\beta$ in $K_\beta$

Similar duality as for finite interval

- No embedding assumption: NP-hard
- Needs some modification
Minimal persistent $d$-cycles of infinite intervals for weak $(d + 1)$-pseudomanifold in $\mathbb{R}^{d+1}$

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• Needs some modification
The problems for which we prove the NP-hardness:

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NP-hardness proof

Suspension: Shifting dimension for reduction

Definition

\[ SK = \{\{\omega_1\}, \{\omega_2\}\} \cup K \cup \left( \bigcup_{\sigma \in K} \{\sigma \cup \{\omega_1\}, \sigma \cup \{\omega_2\}\} \right) \]

E.g., suspension of \( S^1 \Rightarrow S^2 \):

<table>
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<tr>
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<th>( \tilde{H}_0 )</th>
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<th>( \tilde{H}_2 )</th>
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(Figure courtesy of Wikipedia)
Finite interval hardness

Proposition

$PCYC\text{-FIN}_{d-1}$ reduces to $PCYC\text{-FIN}_d$ for $d \geq 2$

Recall: $PCYC\text{-FIN}_1$ is NP-hard

Theorem

$PCYC\text{-FIN}_d$ is NP-hard for $d \geq 1$. 
Infinite interval hardness

Theorem

\( WPCYC-INF_2^+ \) is NP-hard to approximate with any fixed ratio

- \( WPCYC-INF_2^+ \): weights of the simplices are positive
- Reduce from the nearest codeword problem

Theorem

For \( d \geq 2 \), \( WPCYC-INF_d^+ \) is NP-hard to approximate with any fixed ratio

Shift the dimension by suspension
Cosmology
Hurricane/Medical MRI
References


Thank You

(Thanks to NSF for granting a student travel award)